2-Reachability is Almost as Easy as Transitive Closure

ETH zürich

TI Mittagsseminar, November 8, 2016, daniel.graf@inf.ethz.ch

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Connectivity in Directed Graphs

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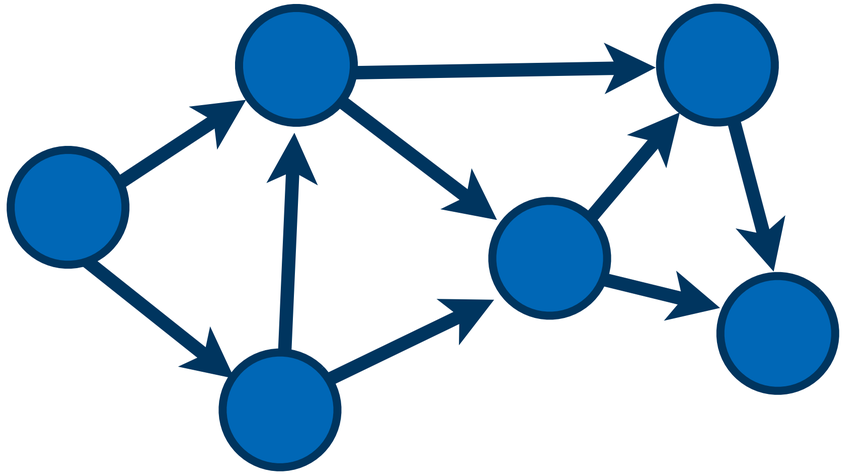

Connectivity in Directed Graphs

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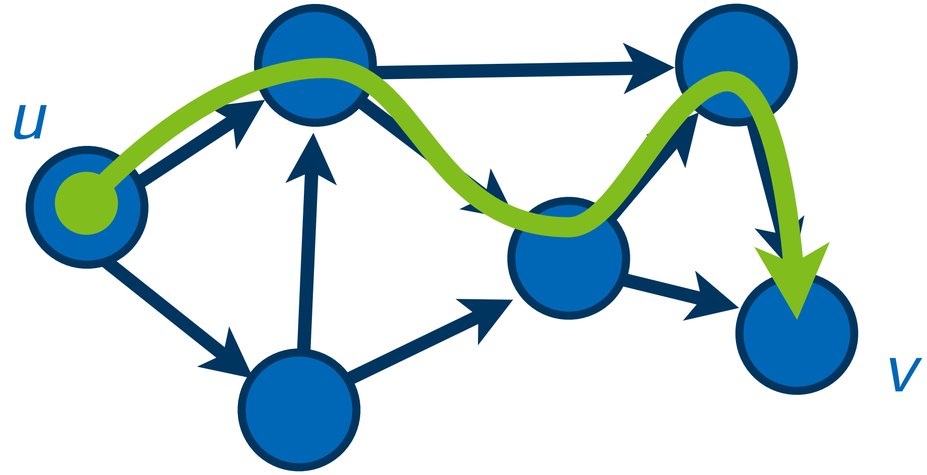
Reachability: can we get from u to v?

Algorithms 101: DFS, toposort, SCC



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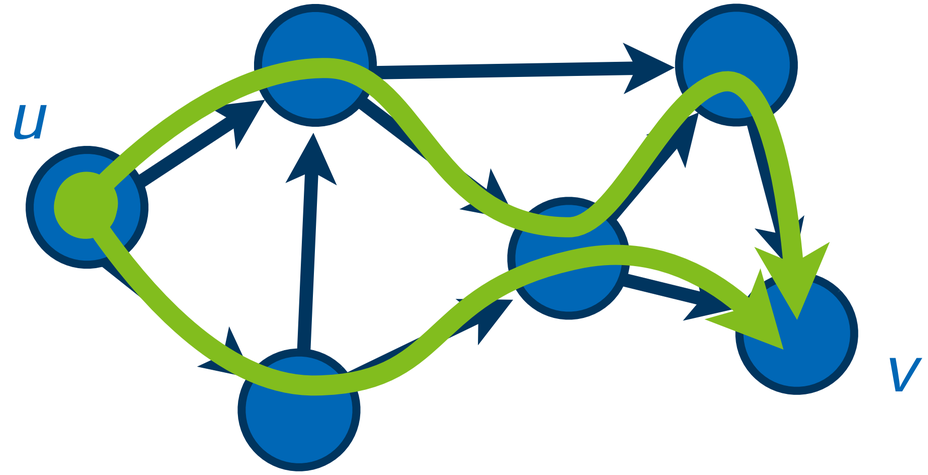
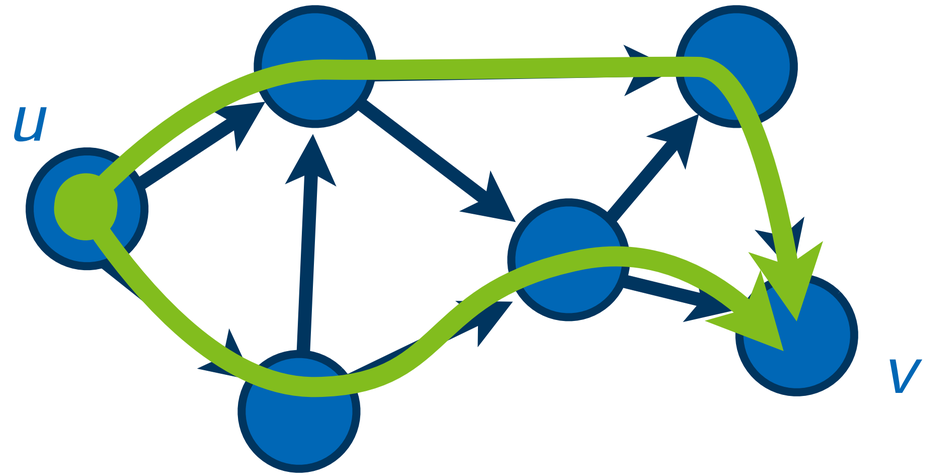
2-Reachability

- two edge-disjoint u-v-path?
- two vertex-disjoint u-v-path?





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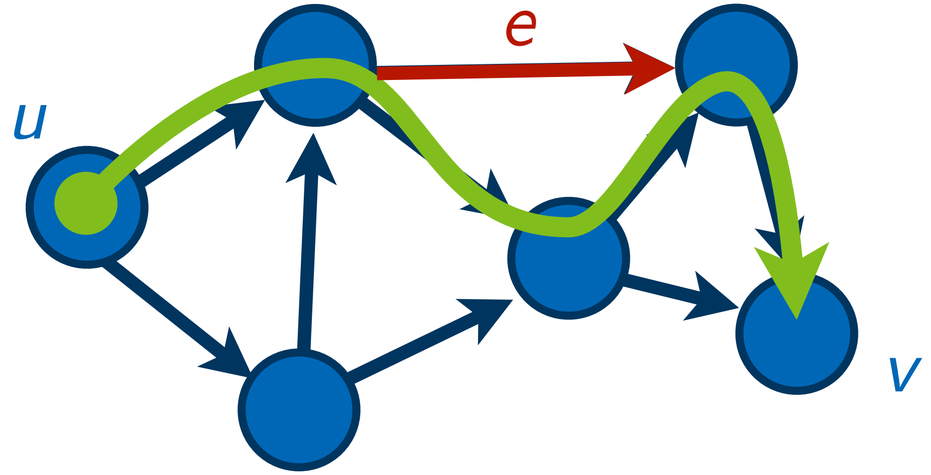
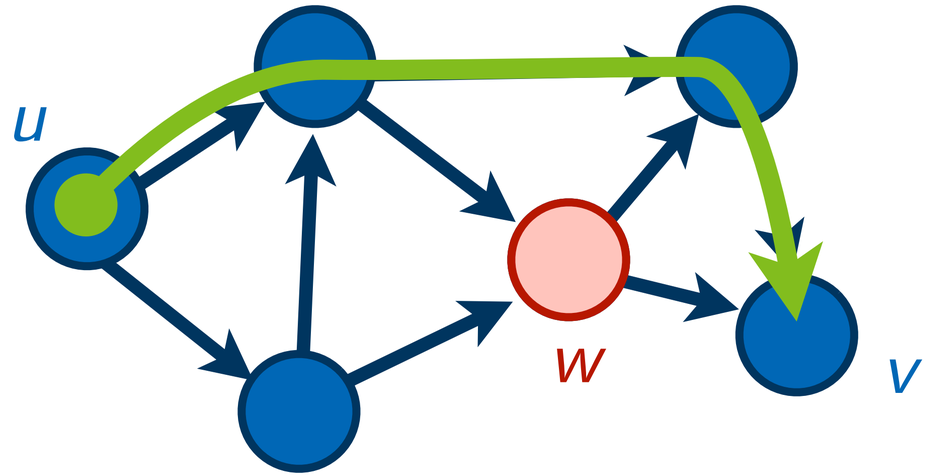
2-Reachability

- two edge-disjoint u-v-path?
- two vertex-disjoint u-v-path?
- u-v-path avoiding e?
- u-v-path avoiding w?





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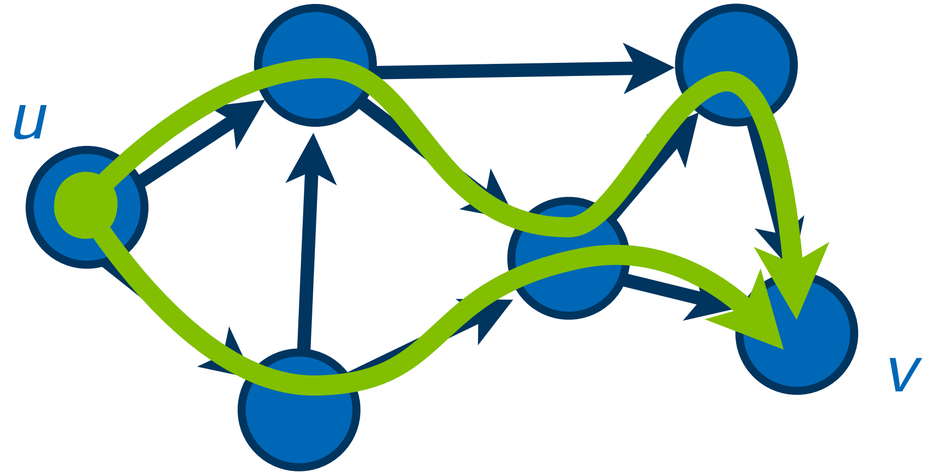

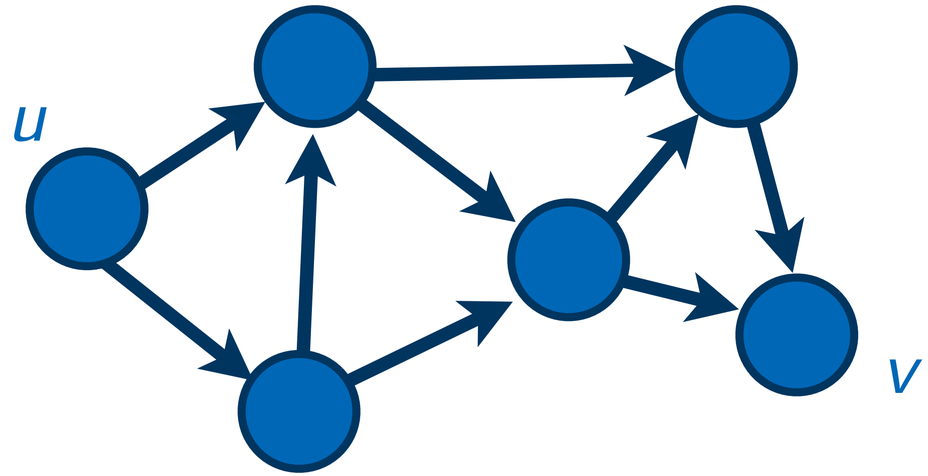

Our Problem: All-Pairs 2-Reachability

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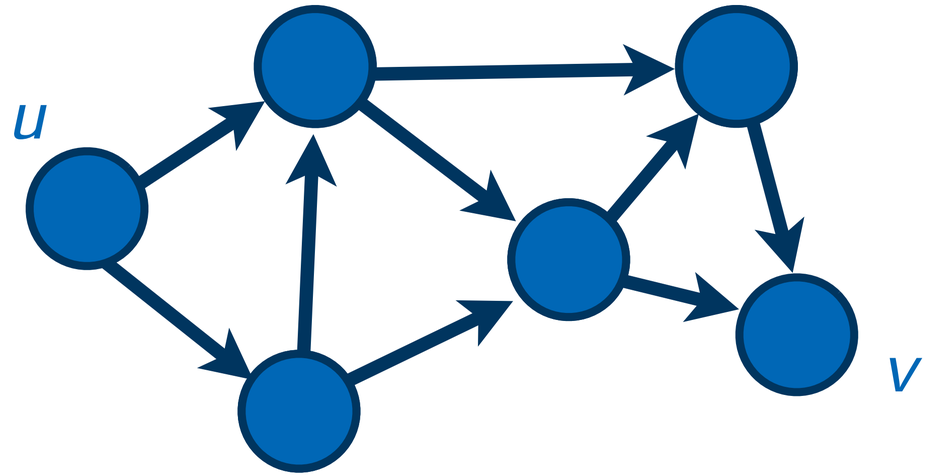
Given: directed graph $G = (V, E)$

with n vertices, m edges



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For all $u, v \in V$, decide if

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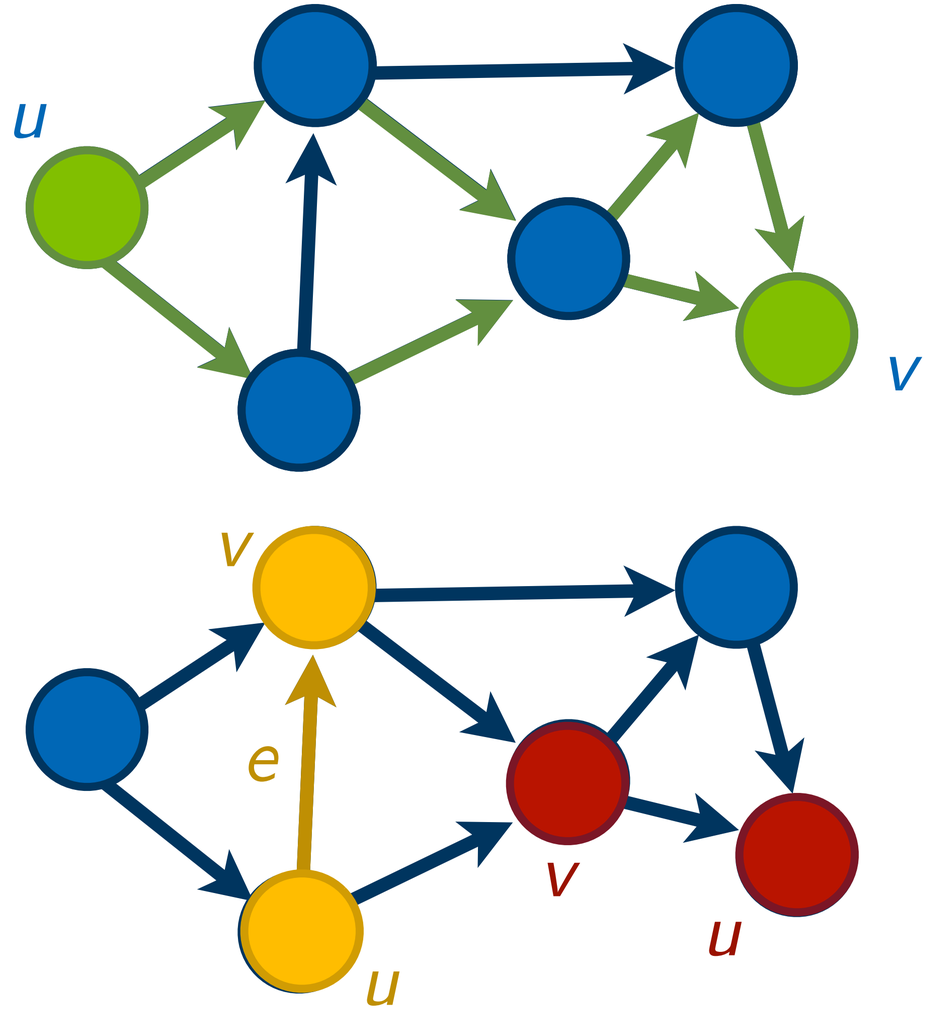
- no u-v-path at all



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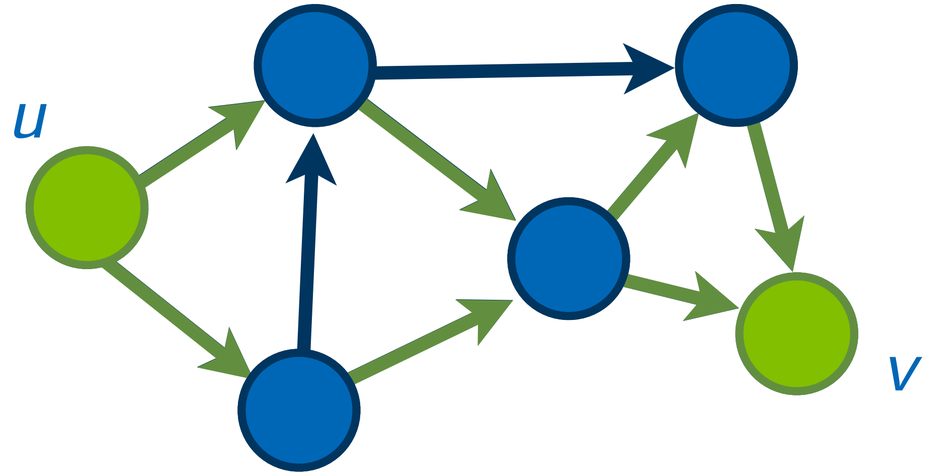

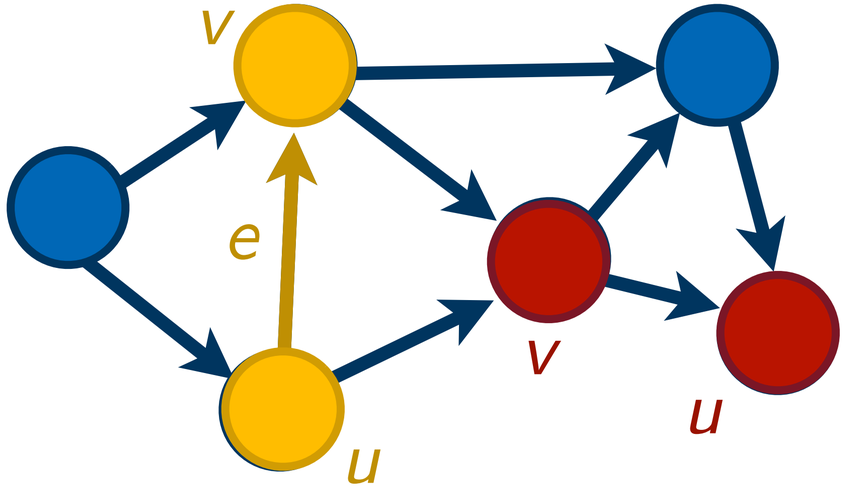

Goal: prepare for constant query time?

2-Reachability / Naive Solution

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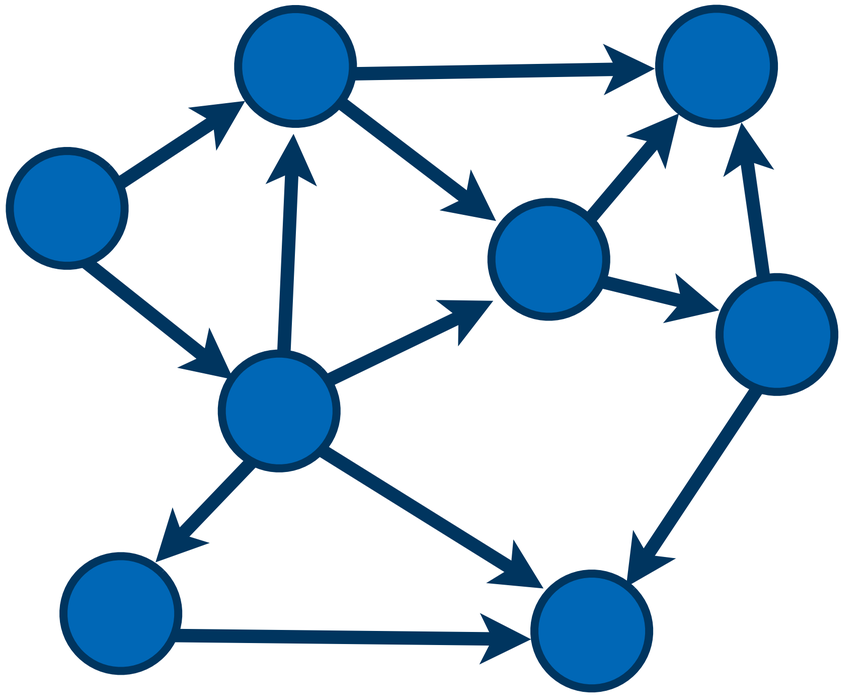
Pseudocode



2-Reachability / Naive Solution

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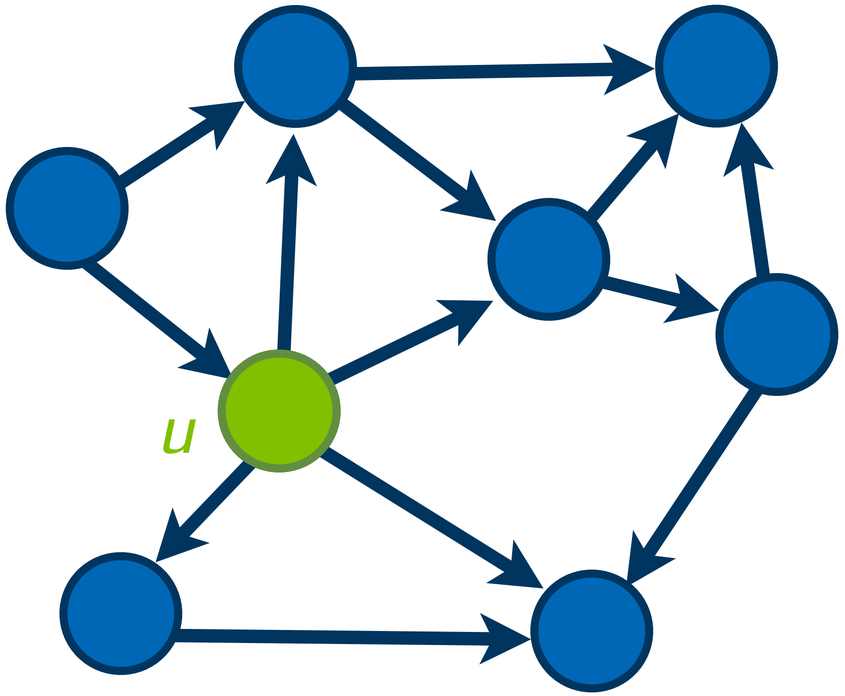
Pseudocode

for every vertex u



2-Reachability / Naive Solution

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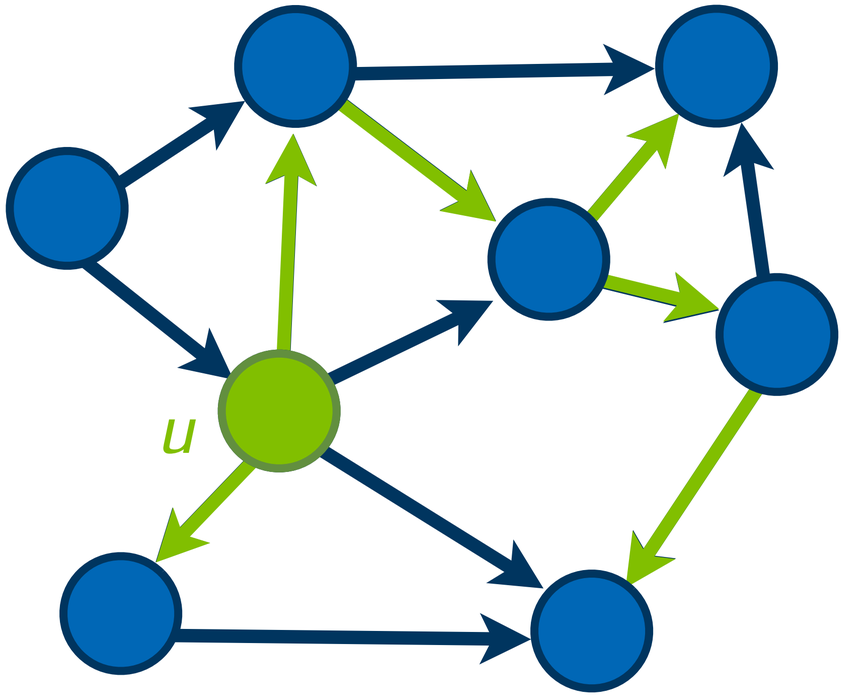
Pseudocode

for every vertex u

    DFS from u

    remember reachability

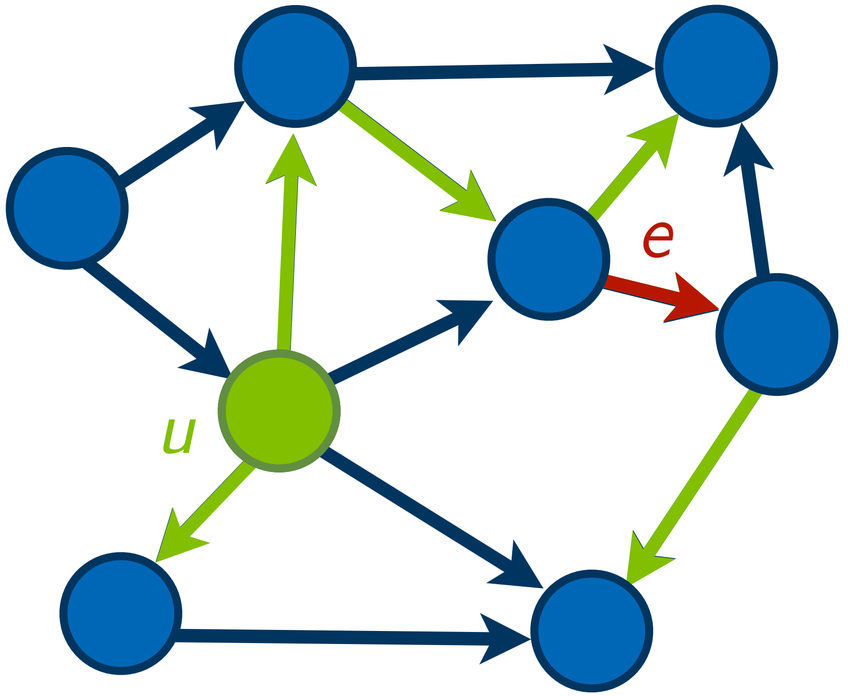


2-Reachability / Naive Solution

ETH zürich

Pseudocode

```
for every vertex u
    DFS from u
    remember reachability
    for every edge e
        
```



2-Reachability / Naive Solution

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        compare reachability
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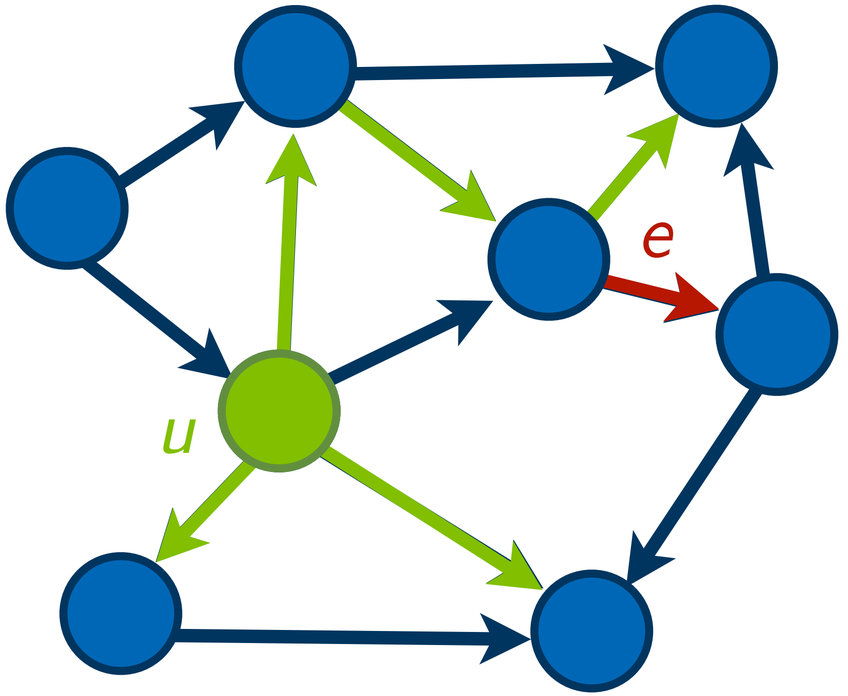

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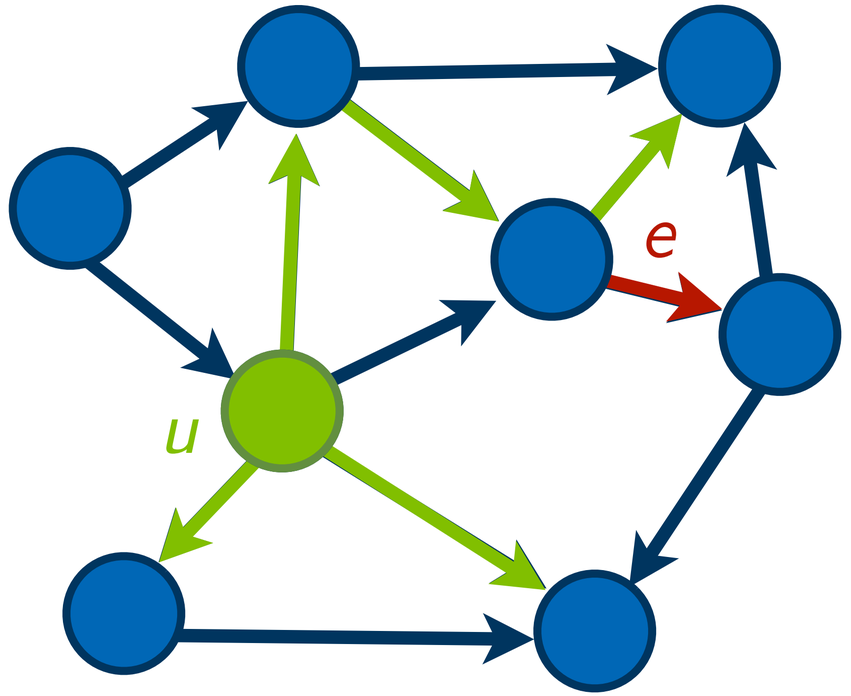

Time: $\mathcal{O}(n \cdot m \cdot m) \subseteq \mathcal{O}(n^5)$

1-Reachability / Transitive Closure

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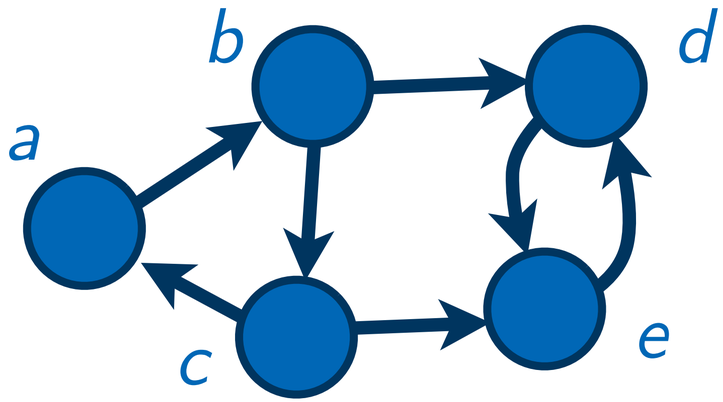

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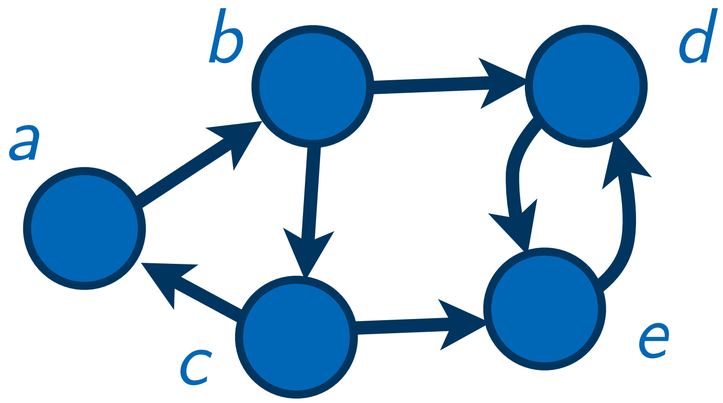

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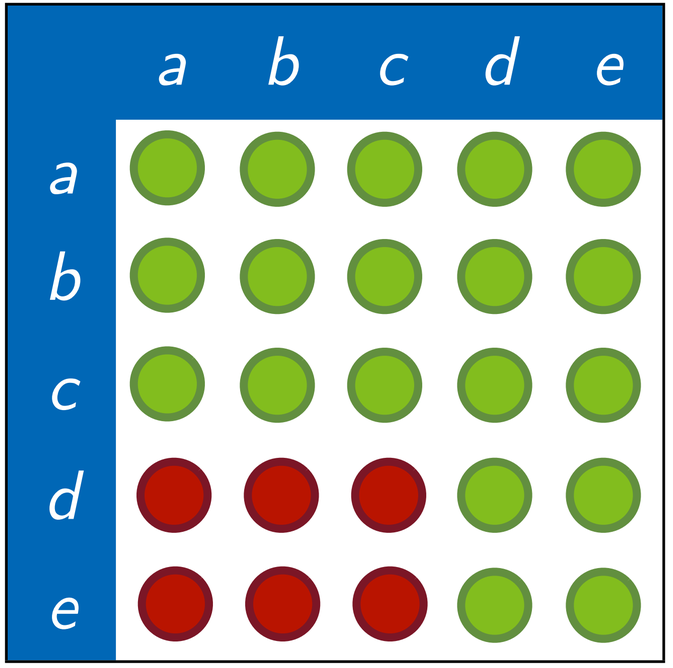

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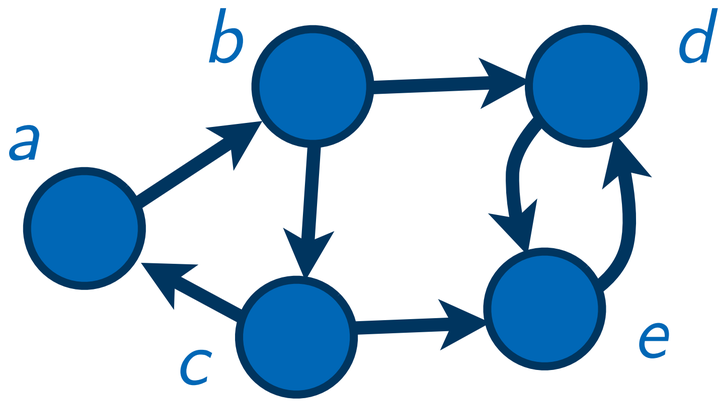

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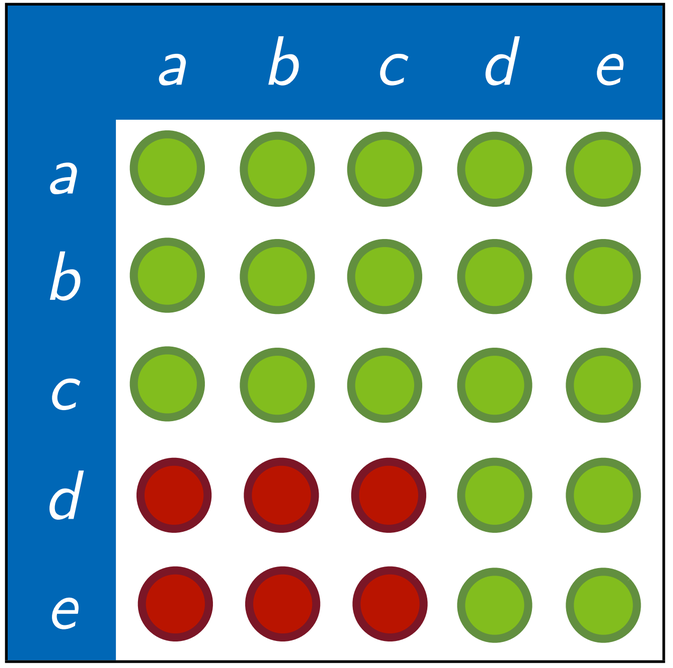

Naive: DFS from every vertex
$\mathcal{O}(n \cdot m) \subseteq \mathcal{O}(n^3)$

Fast Transitive Closure

Fischer, Meyer 1971
Munro, 1971

ETH zürich

Theorem	TC as fast as Boolean matrix multiplication	$\mathcal{O}(n^\omega)$

$\omega < 2.373$

Fast Transitive Closure

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Theorem	TC as fast as Boolean matrix multiplication	$\mathcal{O}(n^\omega)$

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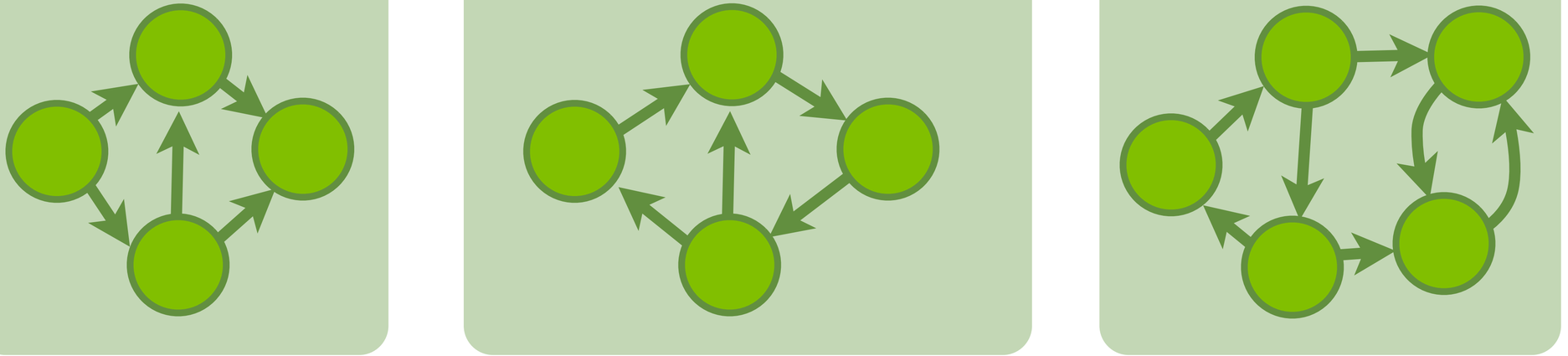
Algorithm:

acyclic + strongly connected + combine



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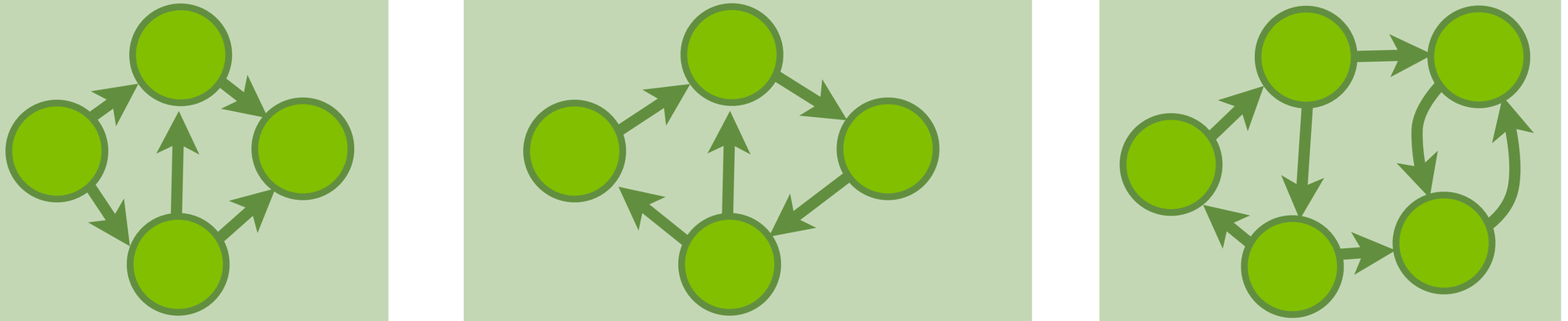

tricky trivial easy

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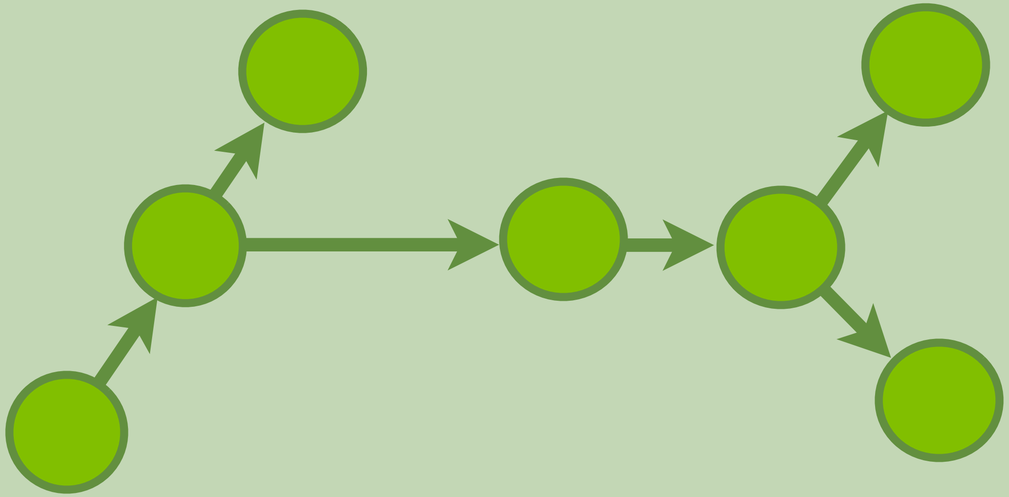

divide & conquer along toposort order

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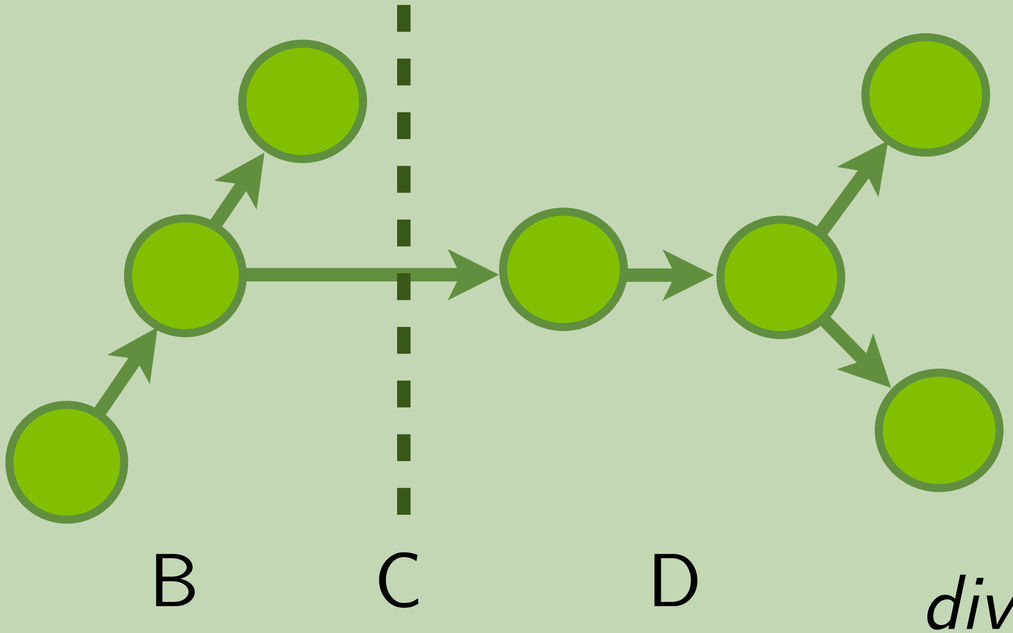

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

divide & conquer along toposort order

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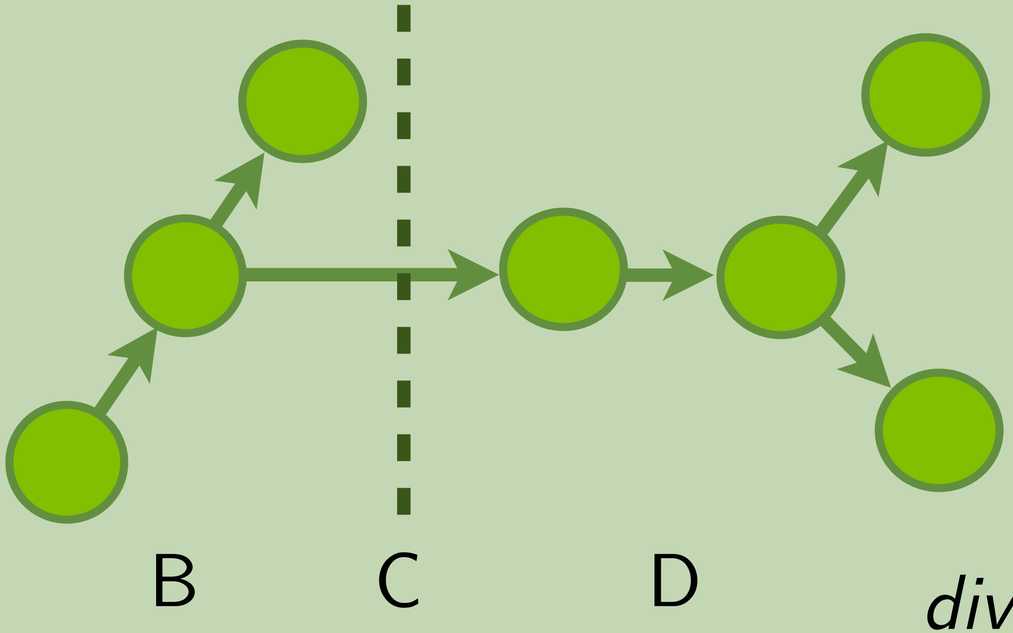

$$A^* = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

divide & conquer along toposort order

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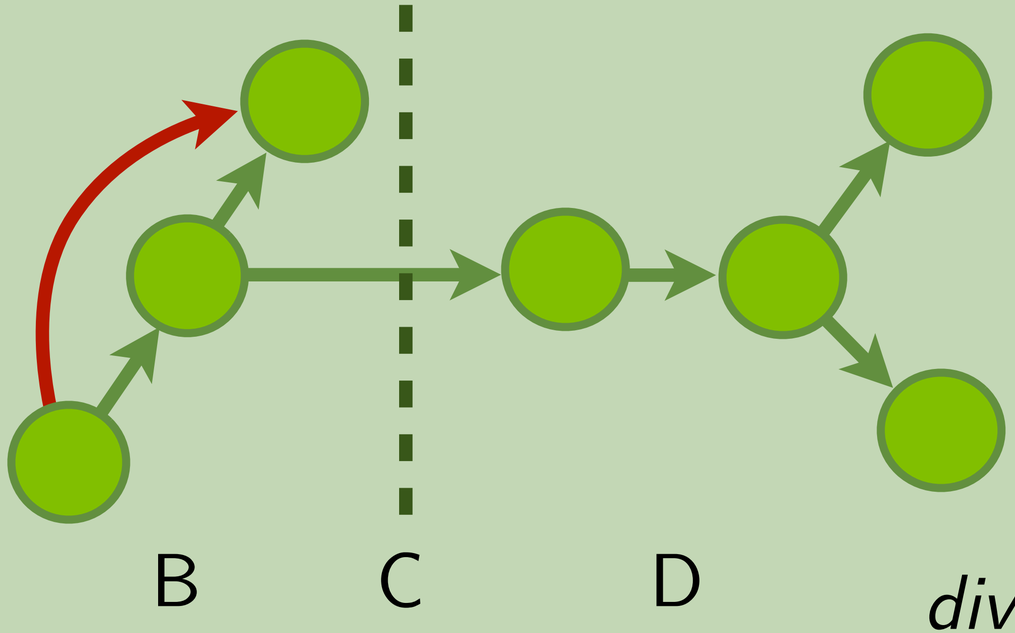

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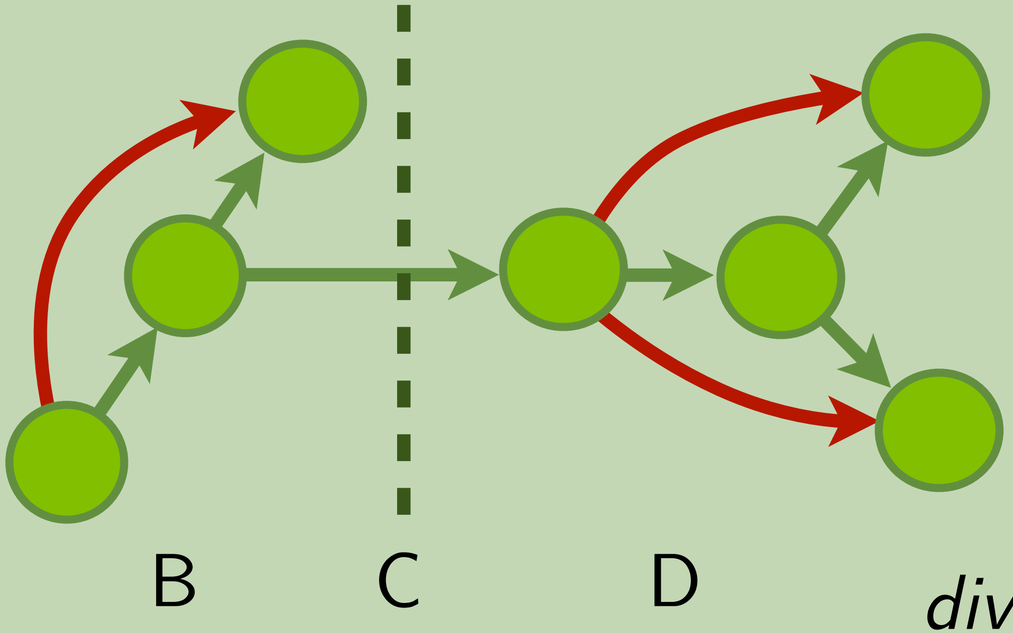

$$A^* = \begin{pmatrix} B^* & C \\ 0 & D^* \end{pmatrix}$$

divide & conquer along toposort order

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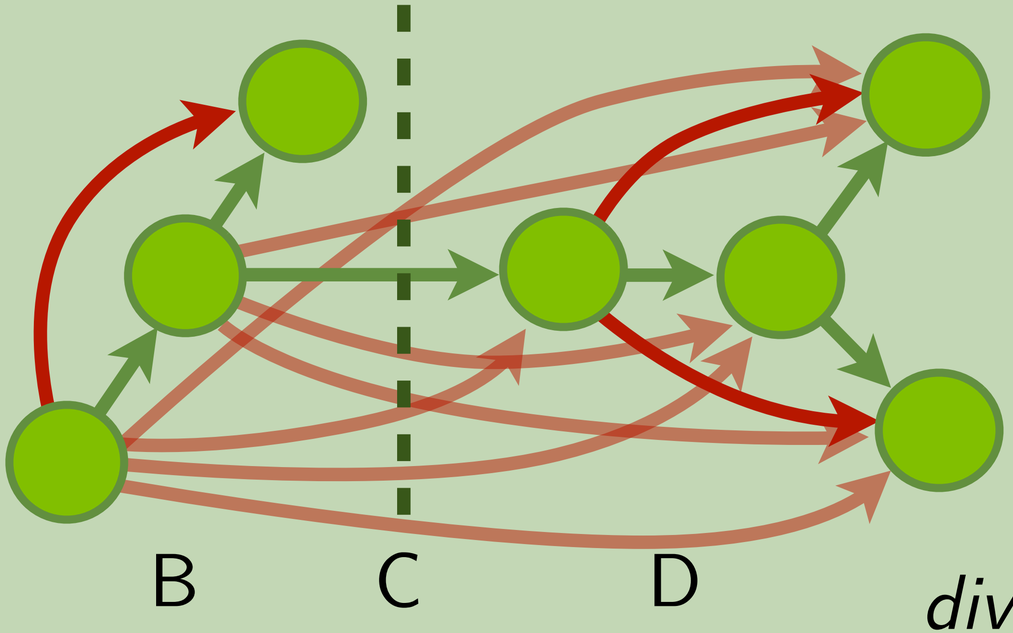

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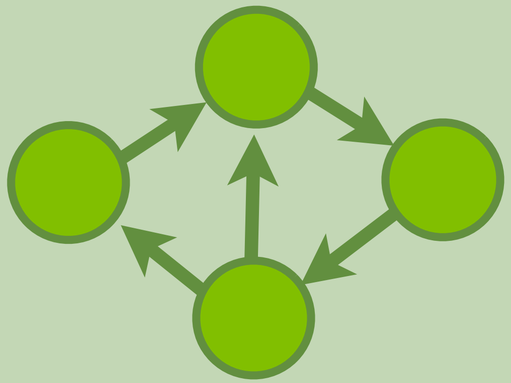
$\mathcal{O}(n^2)$



$A^* = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$

clique

Fast Transitive Closure

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Munro, 1971

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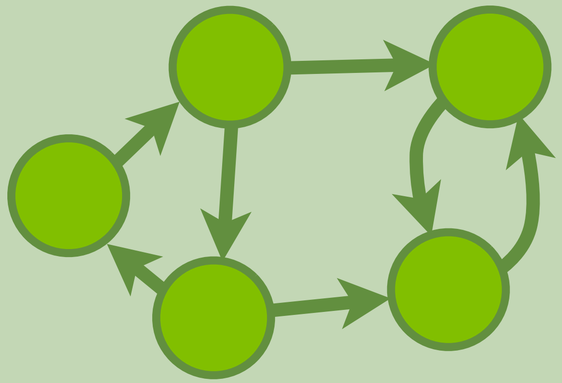
$\mathcal{O}(n^\omega)$



DAG of SCCs

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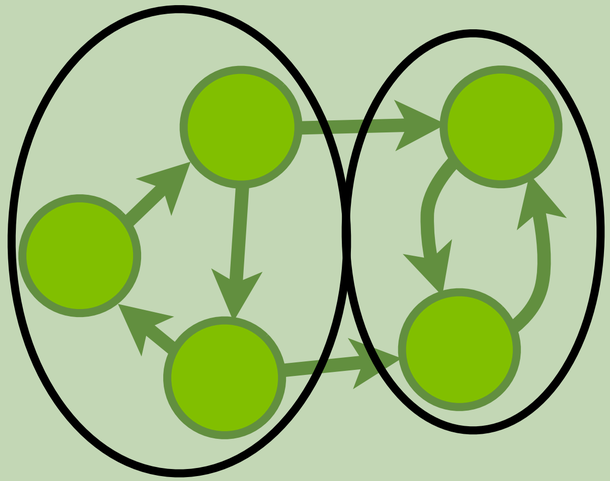

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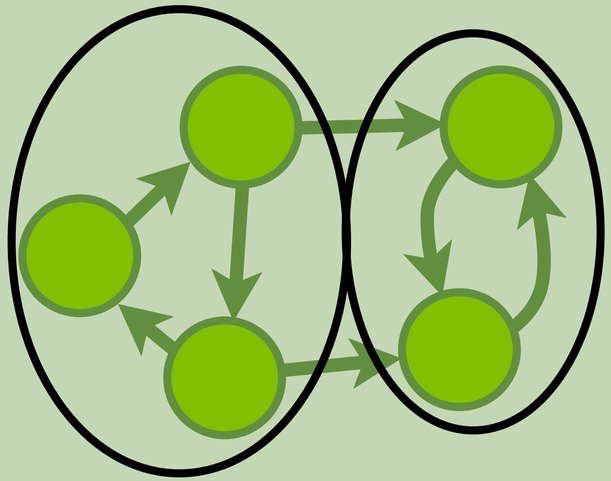
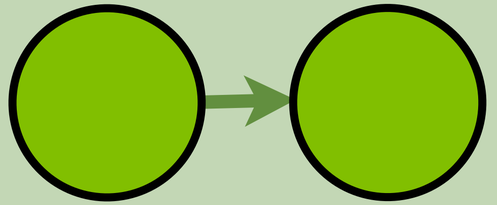
$\mathcal{O}(n^\omega)$





DAG of SCCs

No hope for $o(n^\omega)$

Fischer, Meyer 1971

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Theorem A·B reduces to transitive closure

No hope for $o(n^\omega)$

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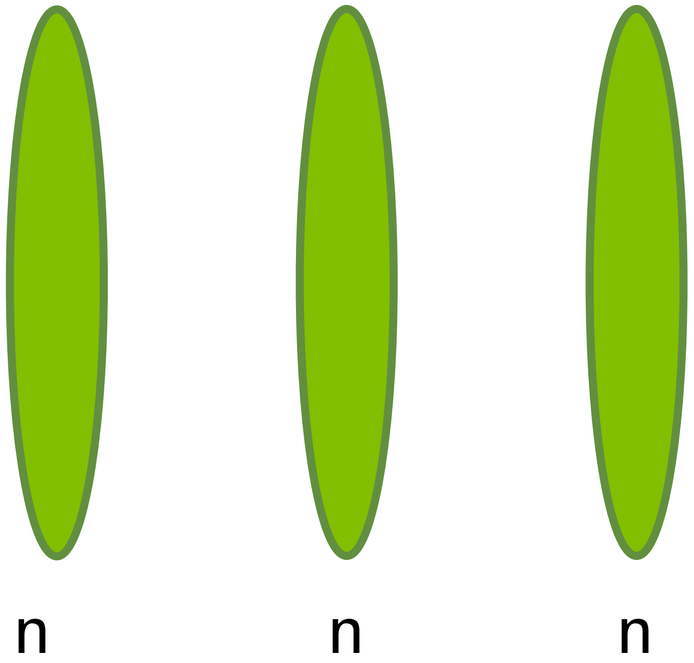
Theorem A·B reduces to transitive closure

Proof:



No hope for $o(n^\omega)$

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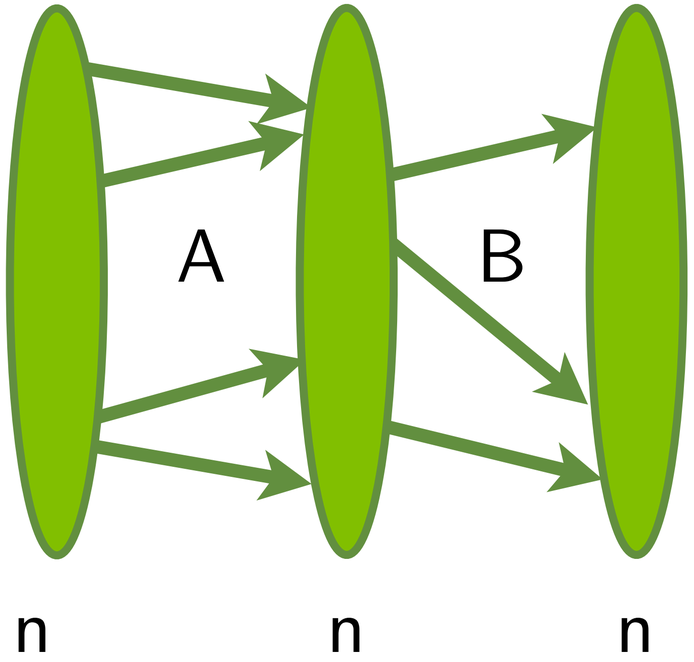

$$\begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}$$

No hope for $o(n^\omega)$

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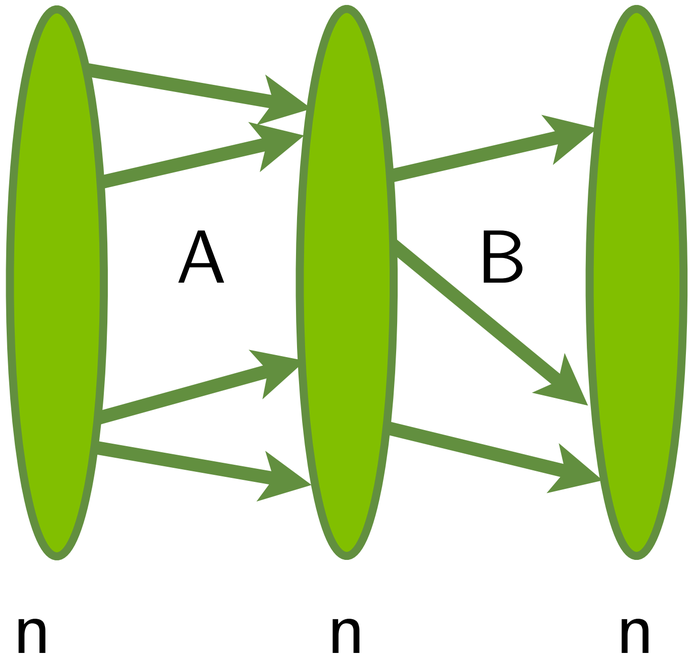
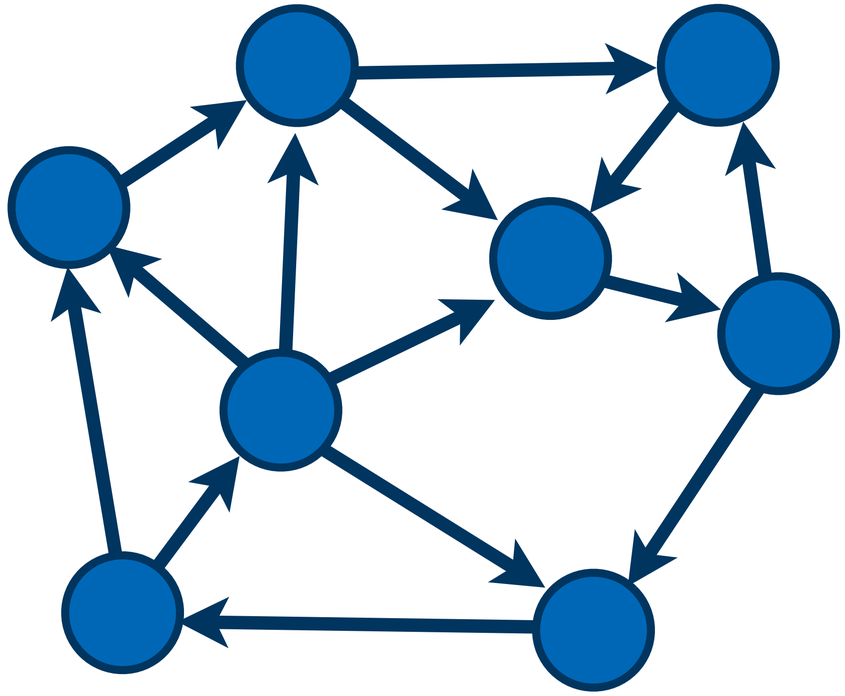


$$\begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}^* = \begin{pmatrix} I & A & AB \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}$$

Improving 2-Reachability?

ETH zürich

Strong bridge: edge whose removal increases number of SCCs



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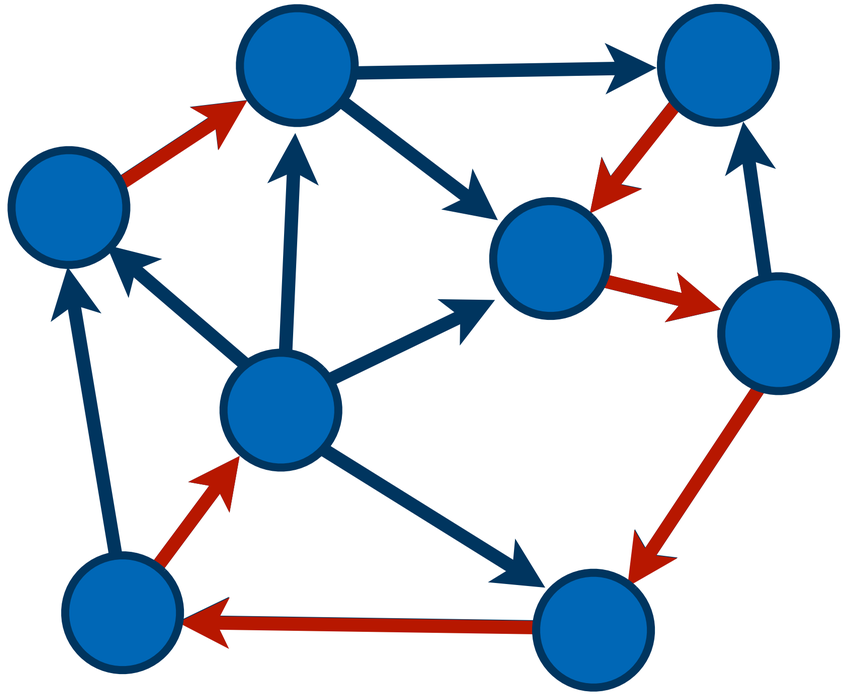

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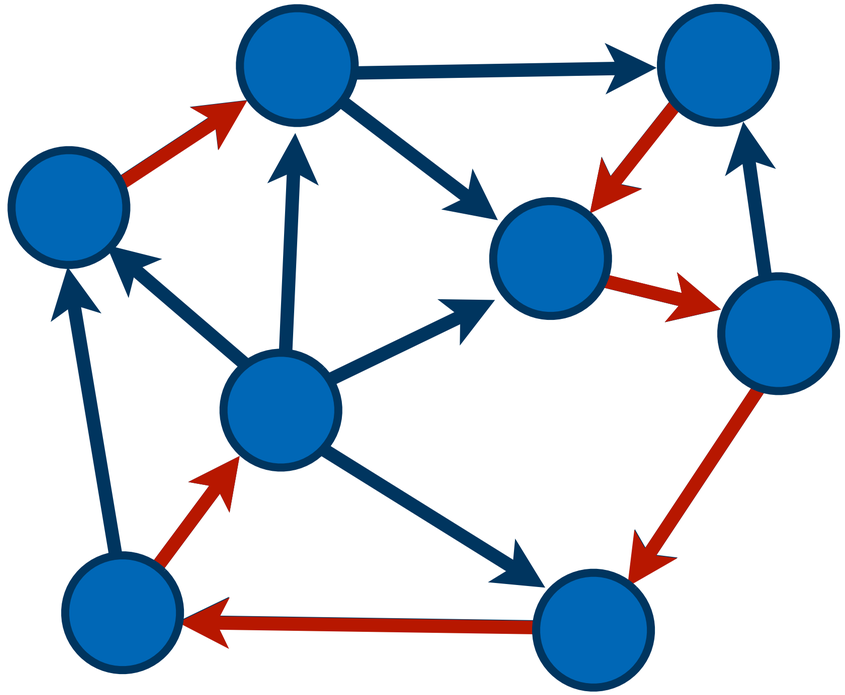
Pseudocode for single SCC

```
for every vertex u
    DFS from u
    for every strong bridge e
        DFS from u in G\e
        compare reachability
```



$$\mathcal{O}(n \cdot n \cdot m) \subseteq \mathcal{O}(n^4)$$

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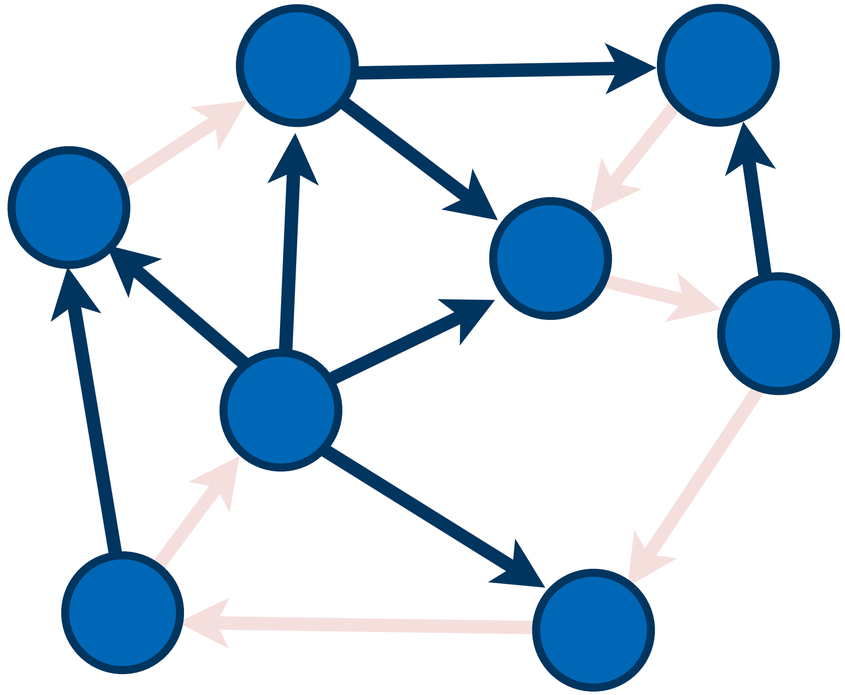
Strong bridge: edge whose removal increases number of SCCs

Pseudocode for single SCC*

for every vertex u
- DFS from u
- ***remove all strong bridges***
- DFS from u in G‘
- compare reachability



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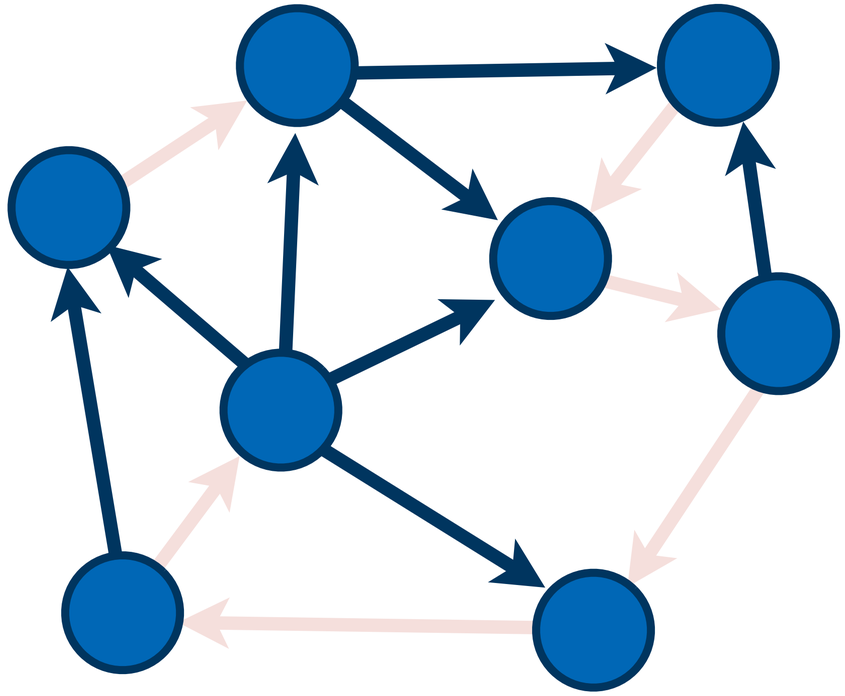

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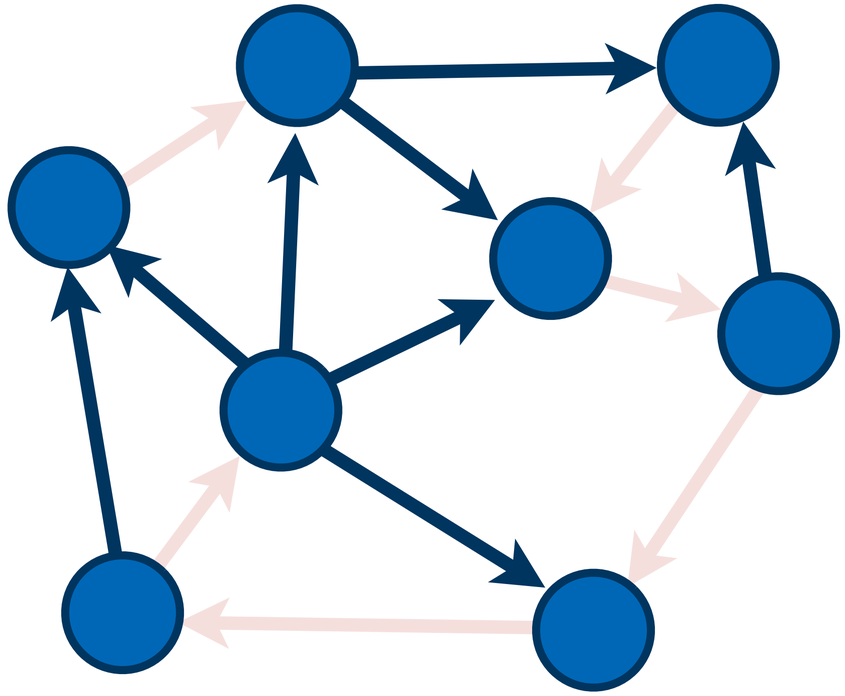

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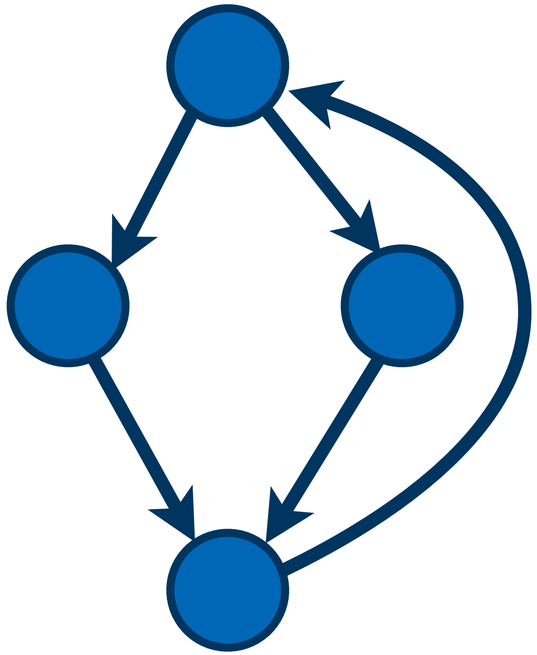

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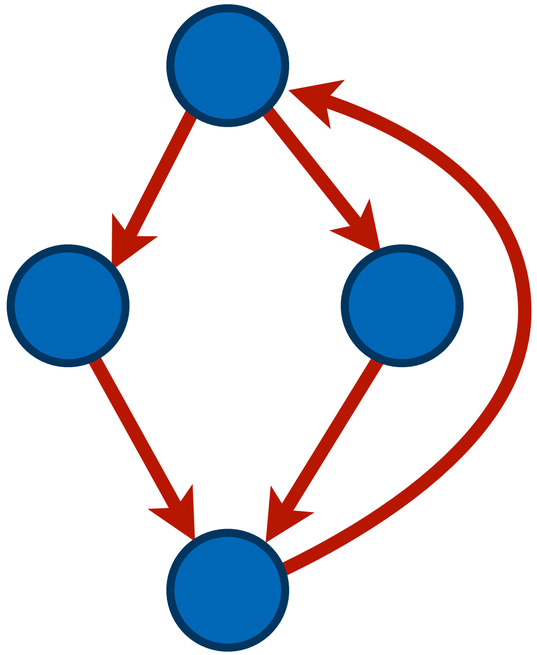

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Previously fastest: single source in $\mathcal{O}(m)$, all pairs in $\mathcal{O}(n \cdot m) \subseteq \mathcal{O}(n^3)$

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[Alstrup, Harel, Lauridsen, Thorup 1990] ← non-trivial combinatorics, domination trees

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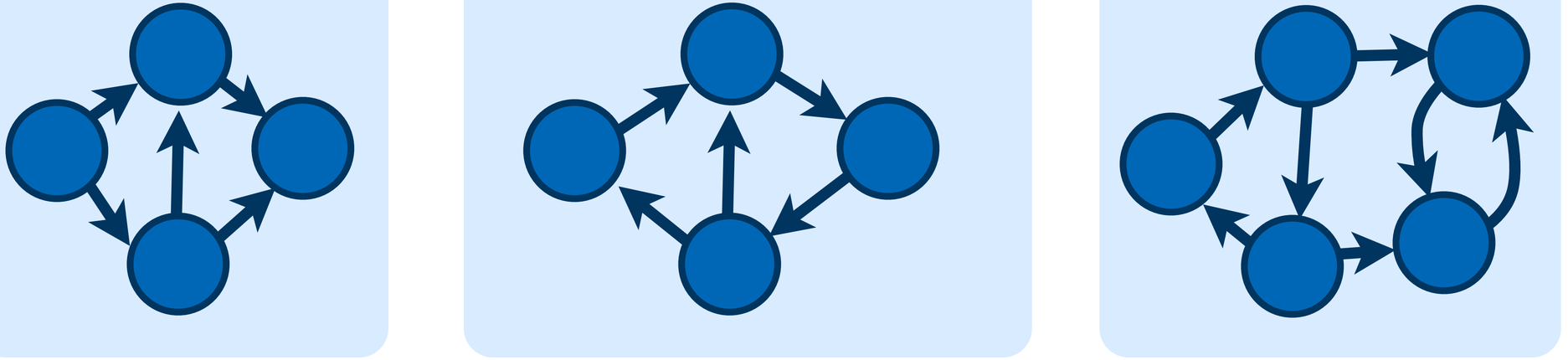
acyclic + strongly connected + combine



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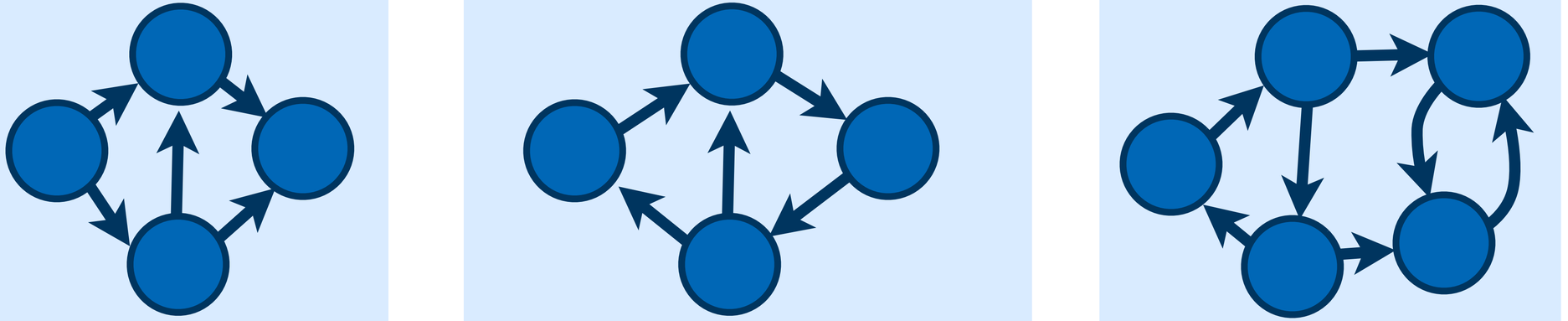

tricky + tricky + non-trivial

State: $L[u,v] = \begin{cases} \bot & \text{no } u\text{-}v\text{-path} \\ e & \text{first common edge of all } u\text{-}v\text{-paths} \\ \top & \text{multiple disjoint } u\text{-}v\text{-paths} \end{cases}$

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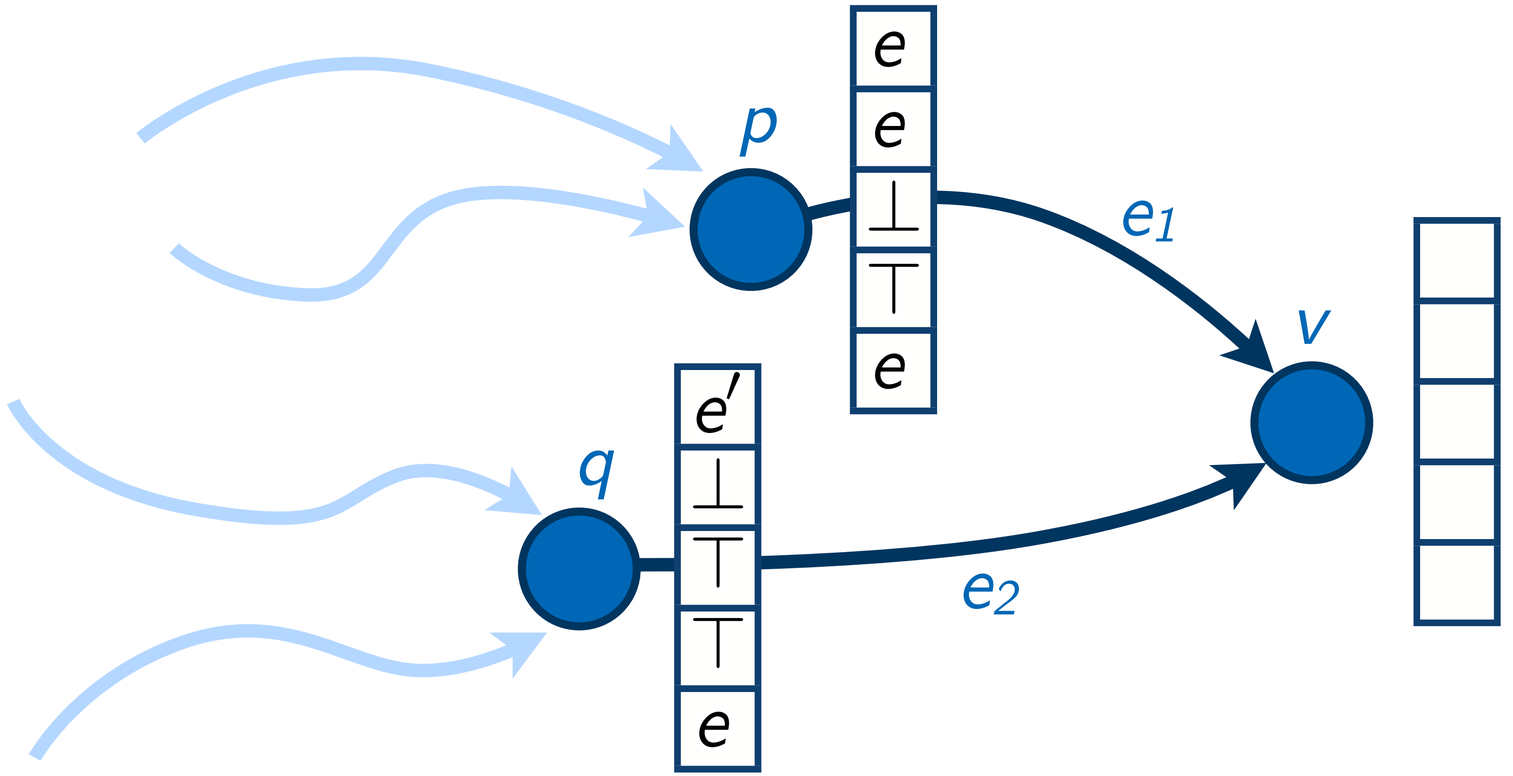

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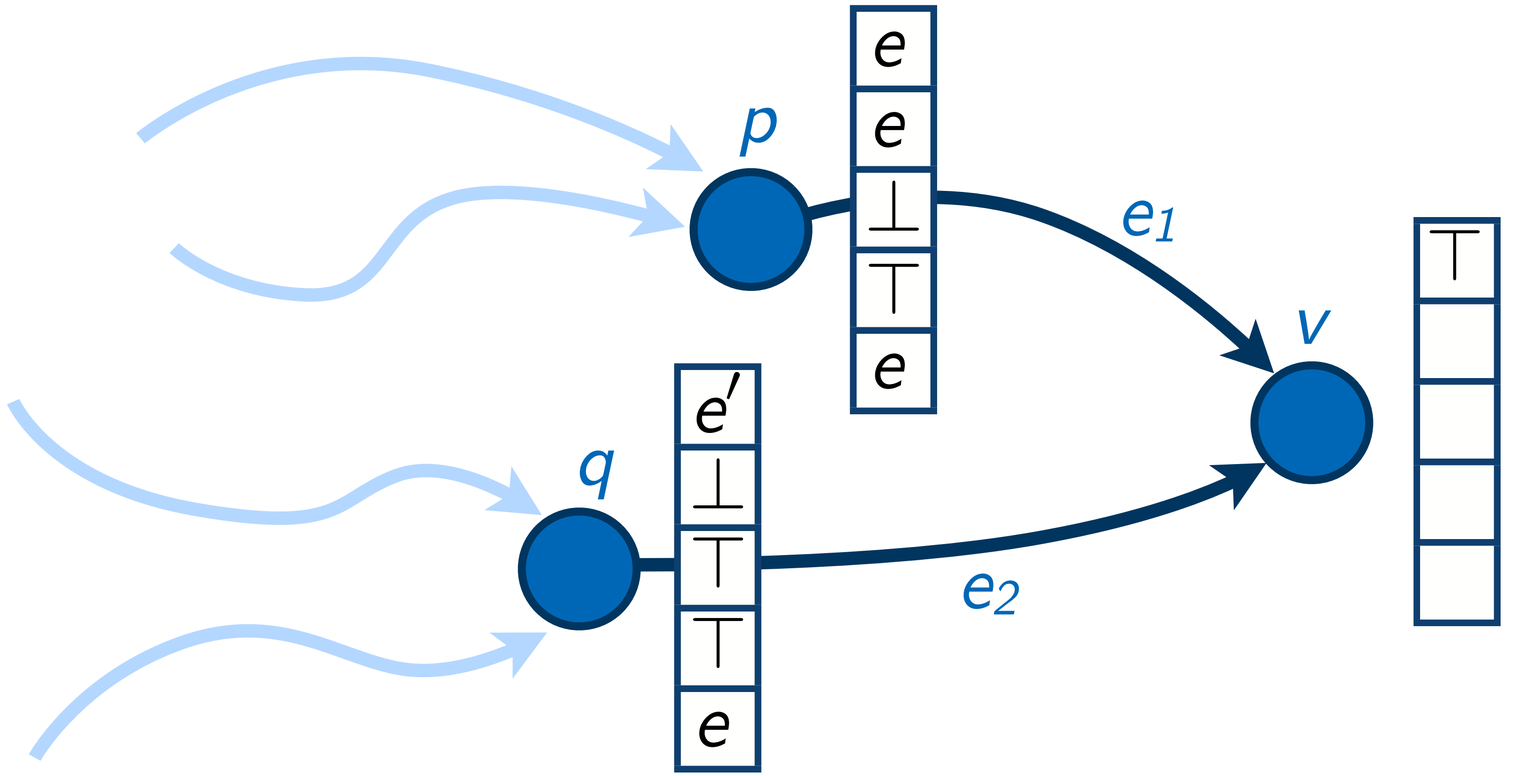

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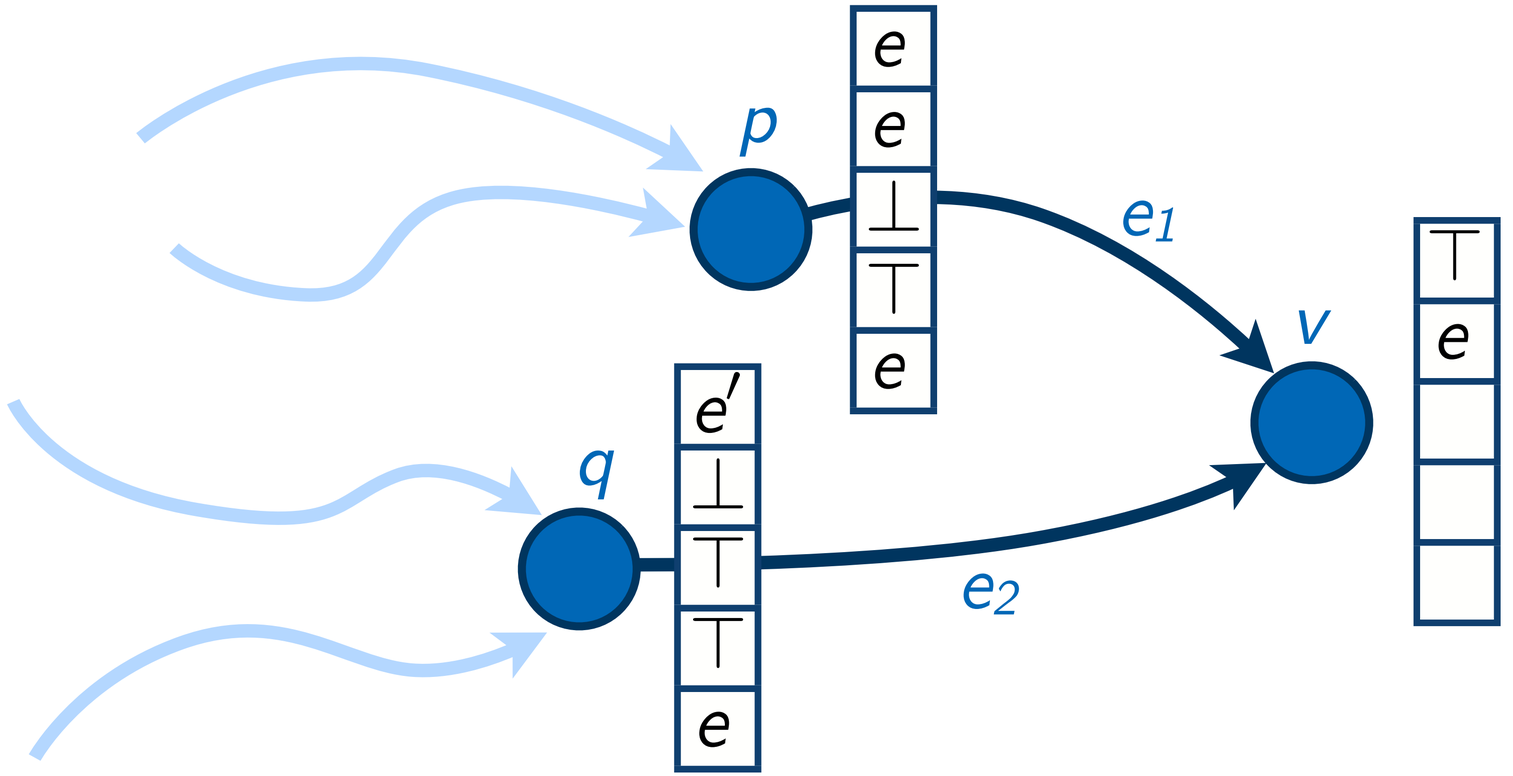

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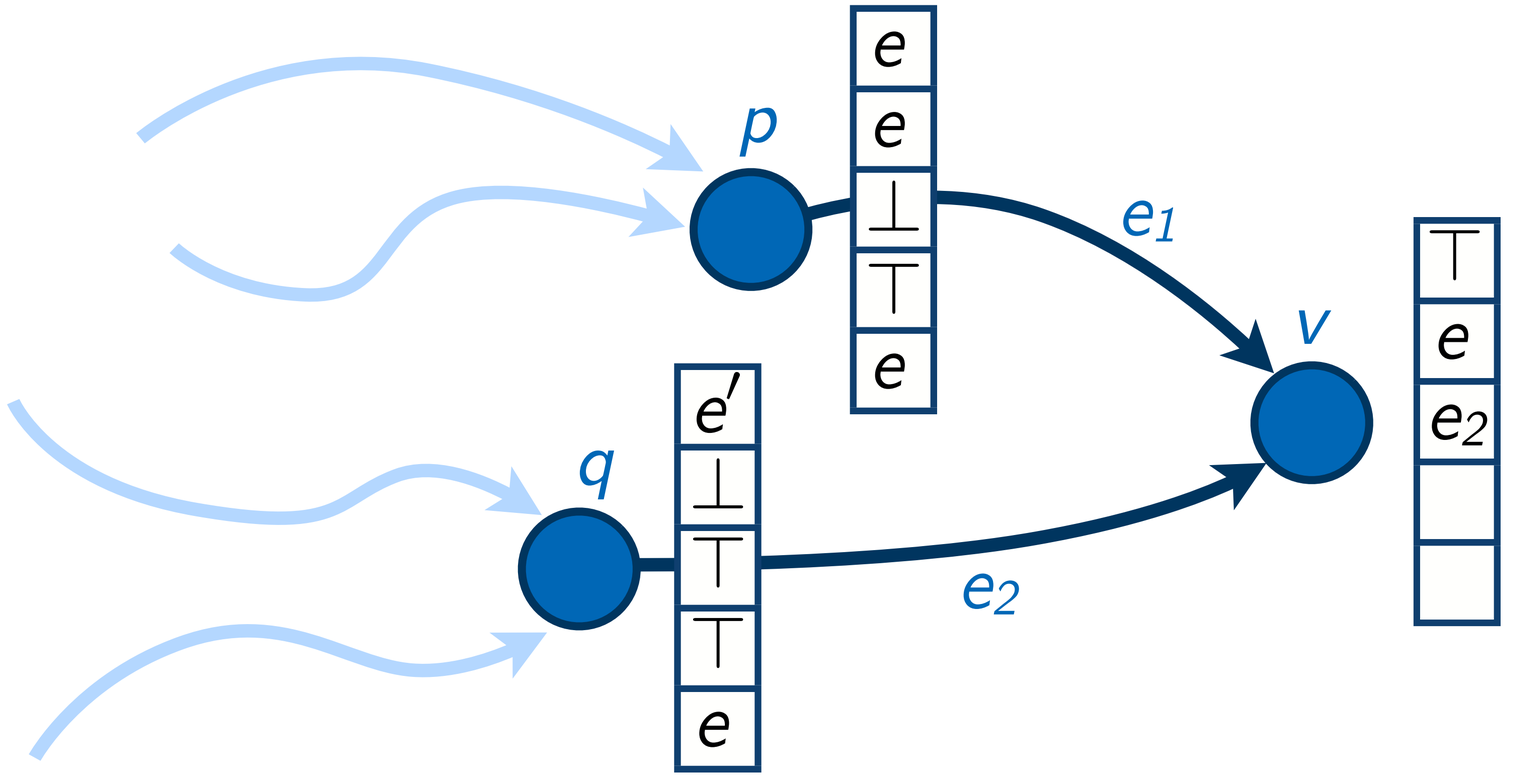

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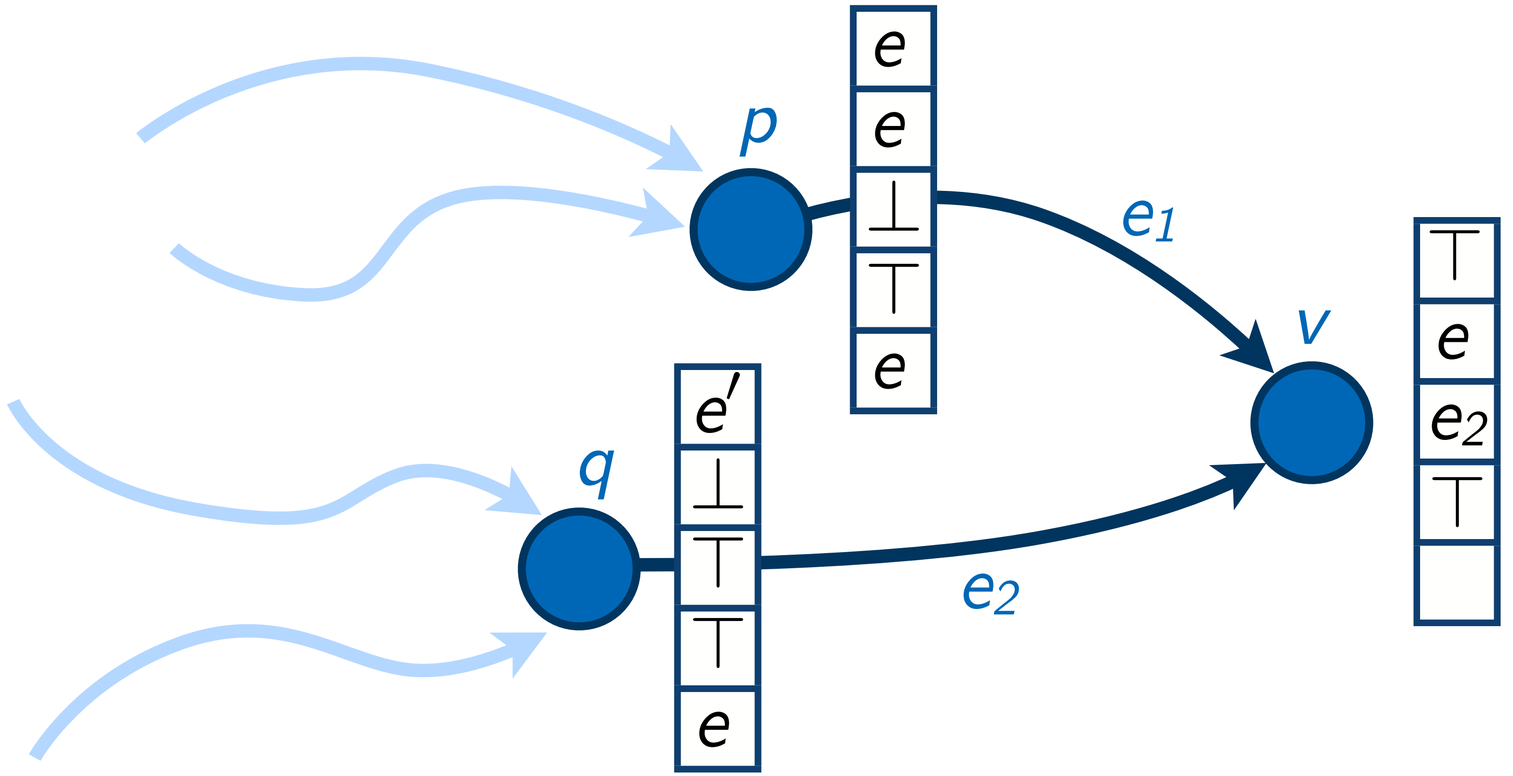

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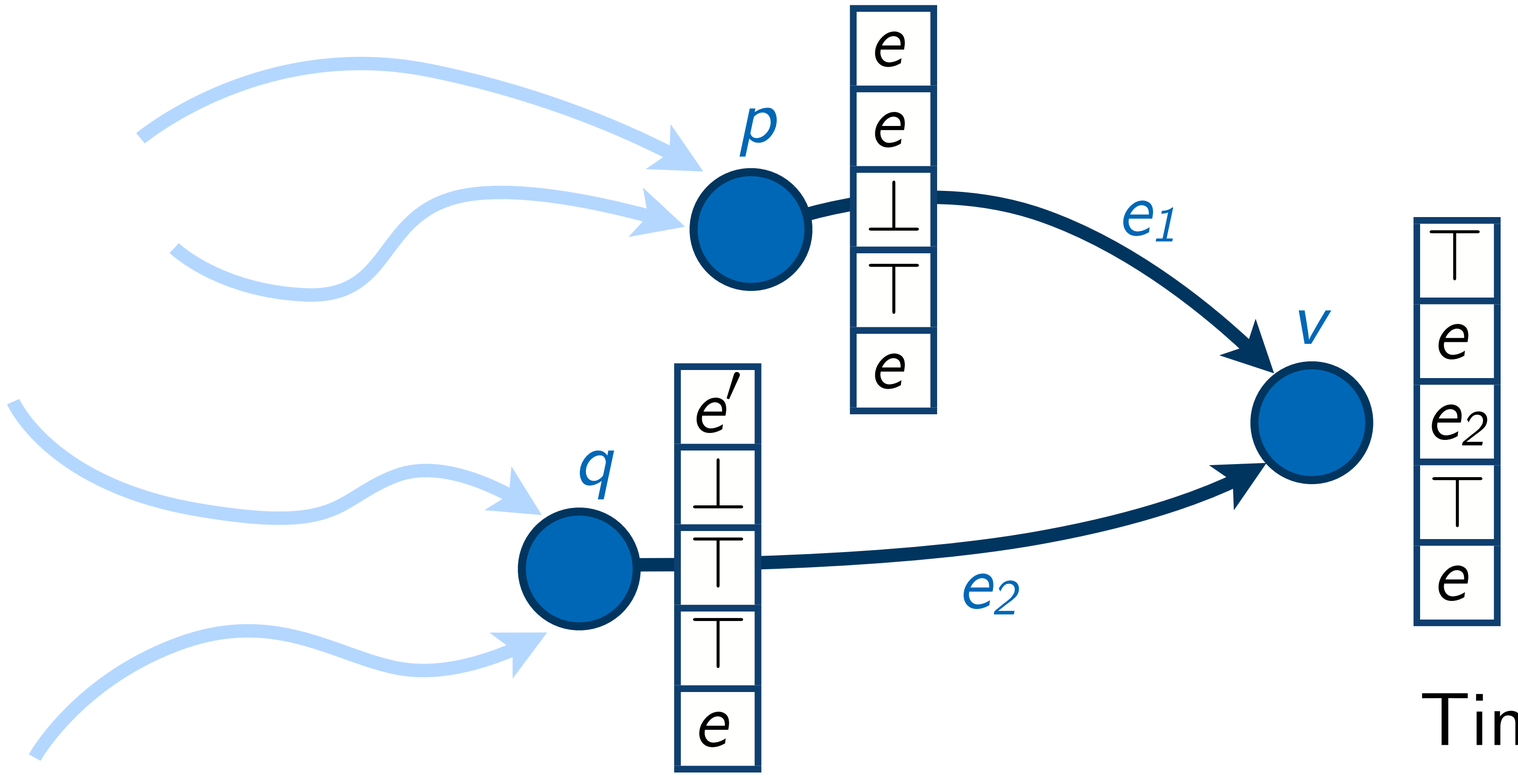
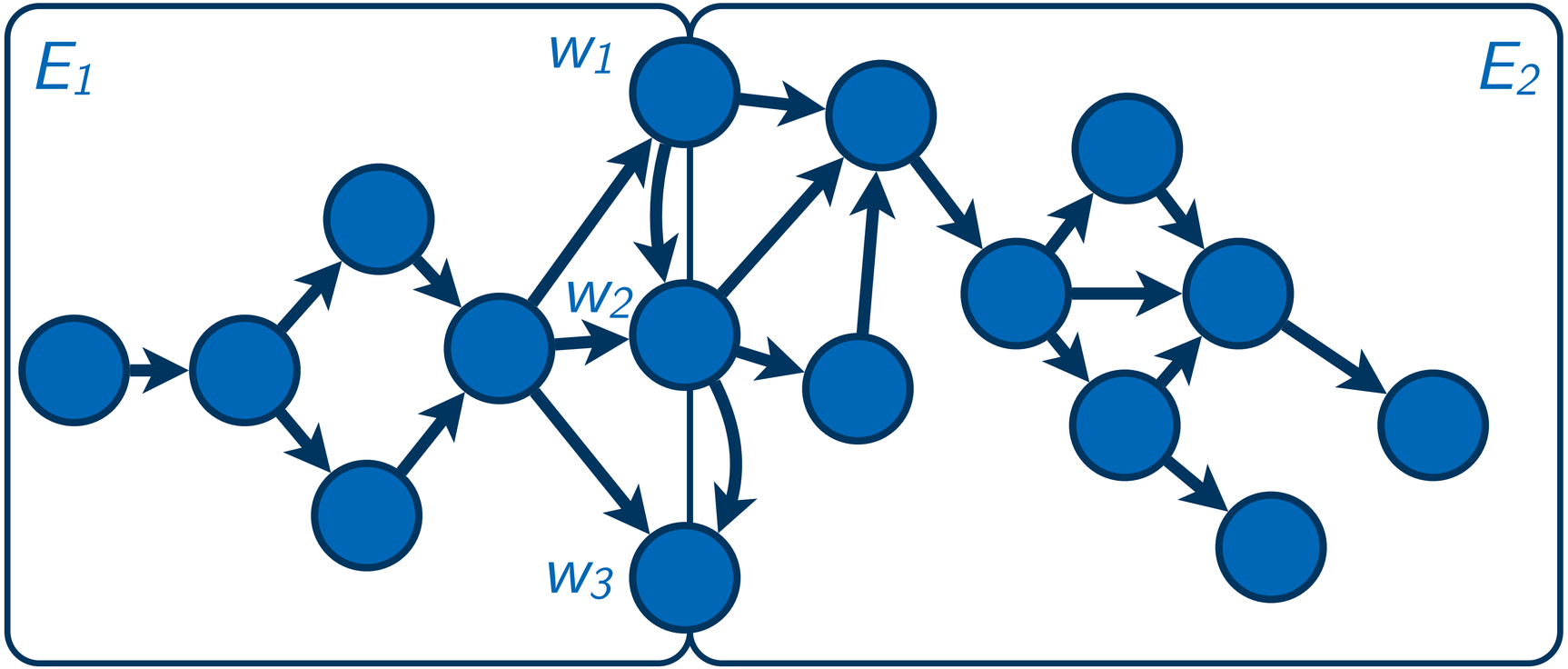
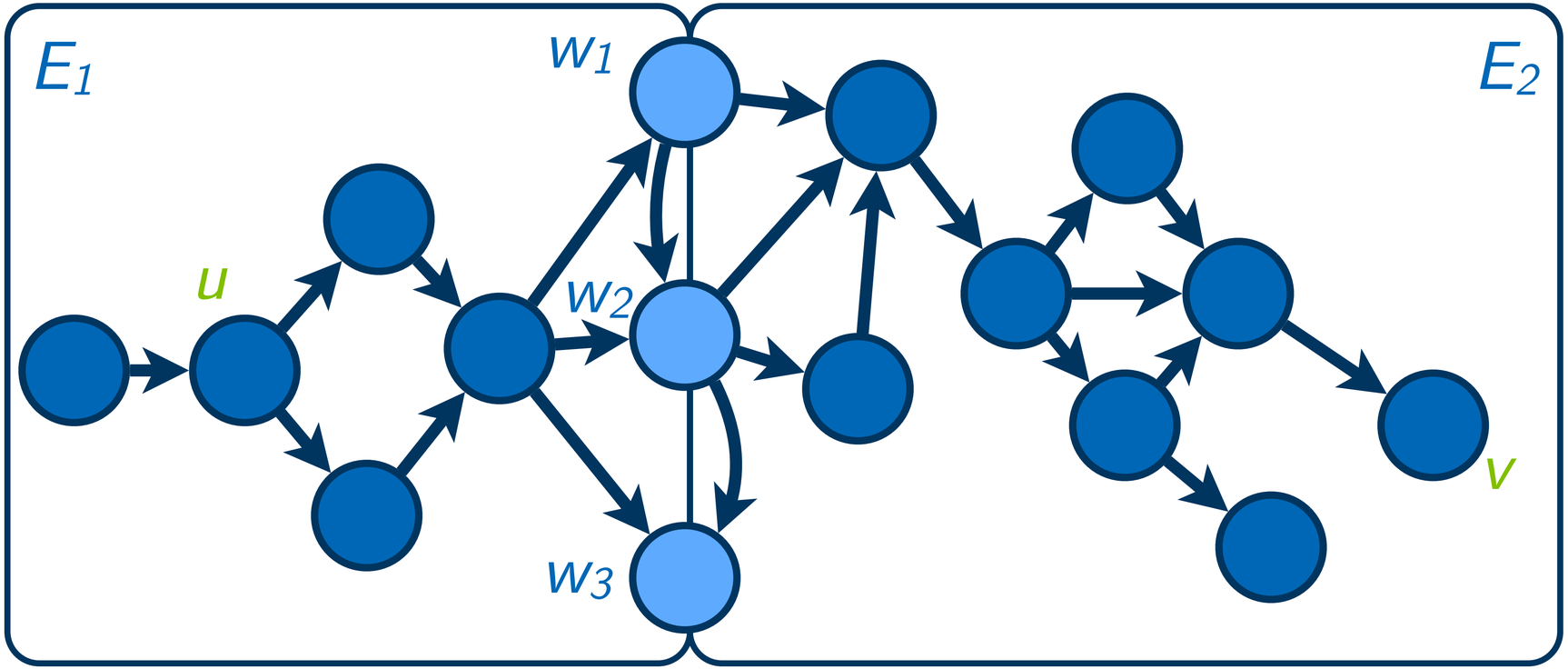


Time: $\mathcal{O}(n \cdot m) \subseteq \mathcal{O}(n^3)$

Edge split: $E = E_1 \dot\cup E_2$ once you leave E_1, you can never go back



Path families:



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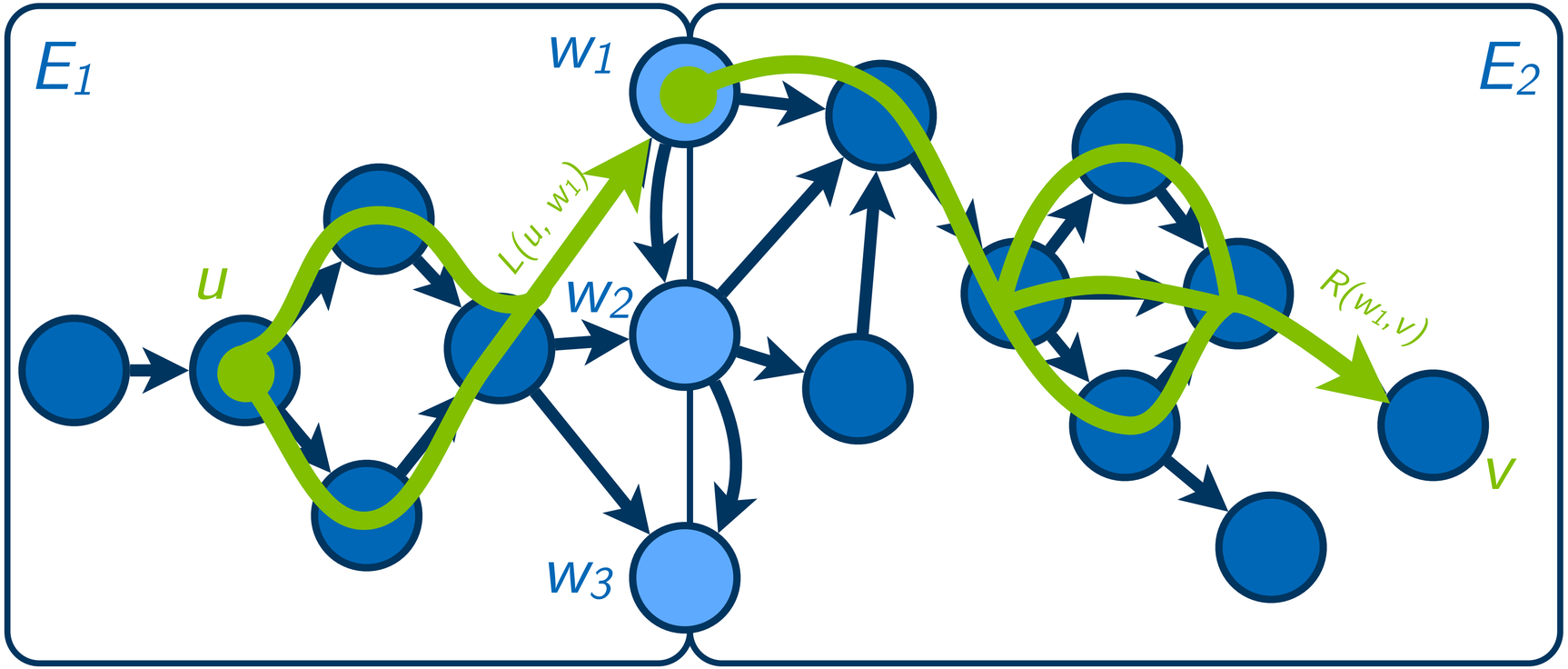

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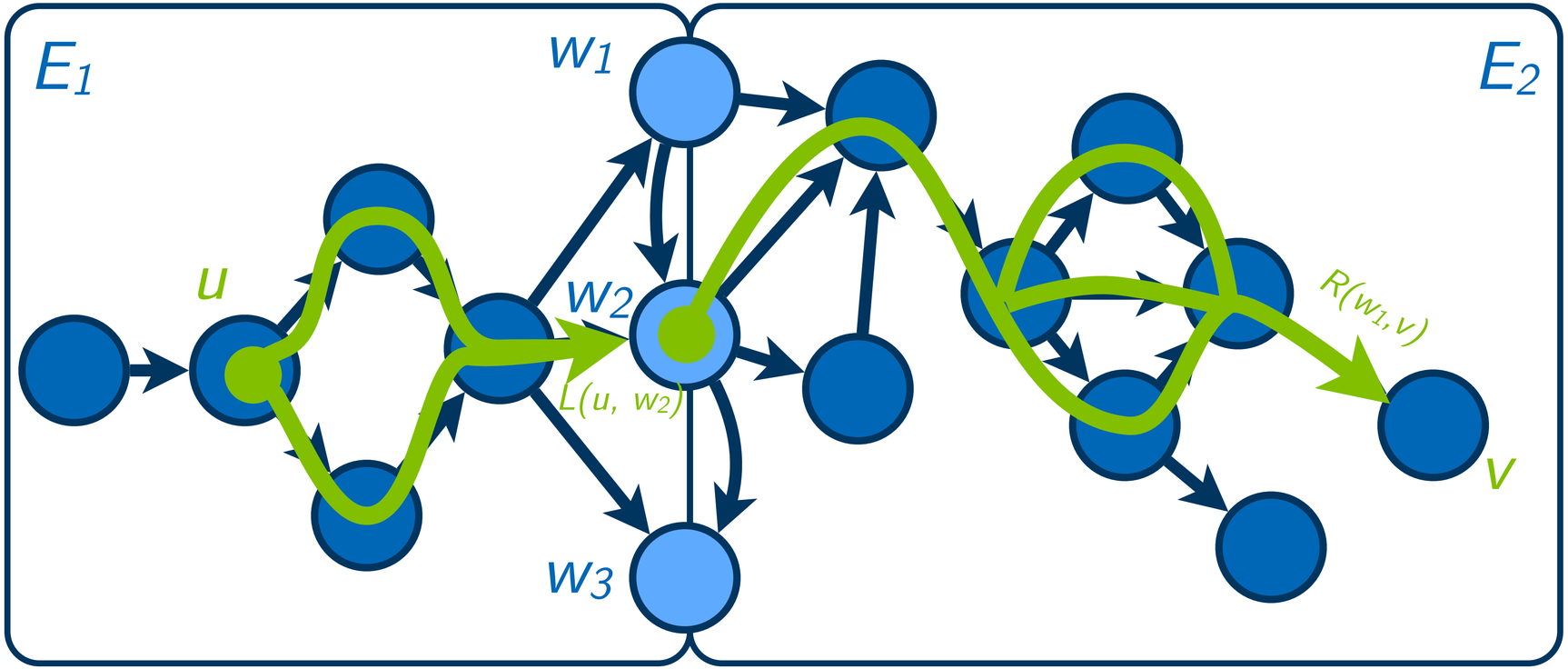

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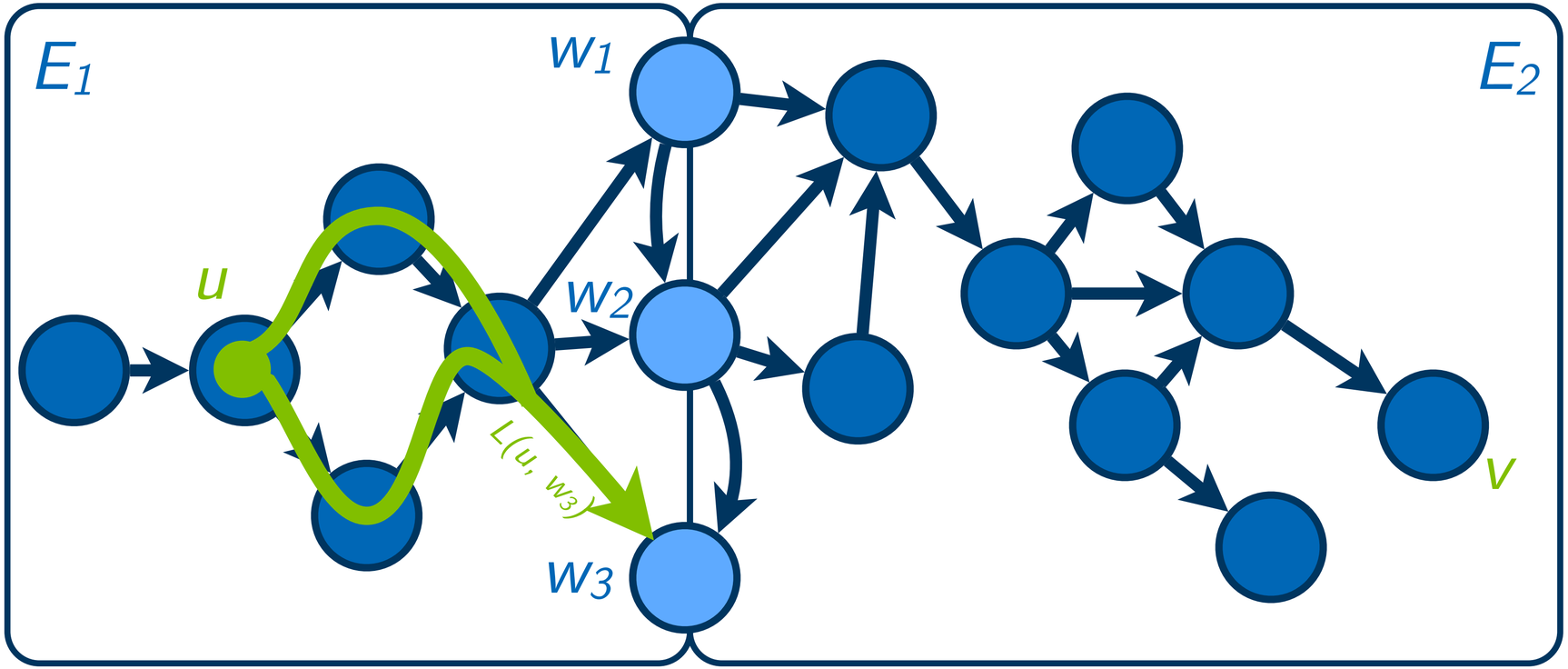

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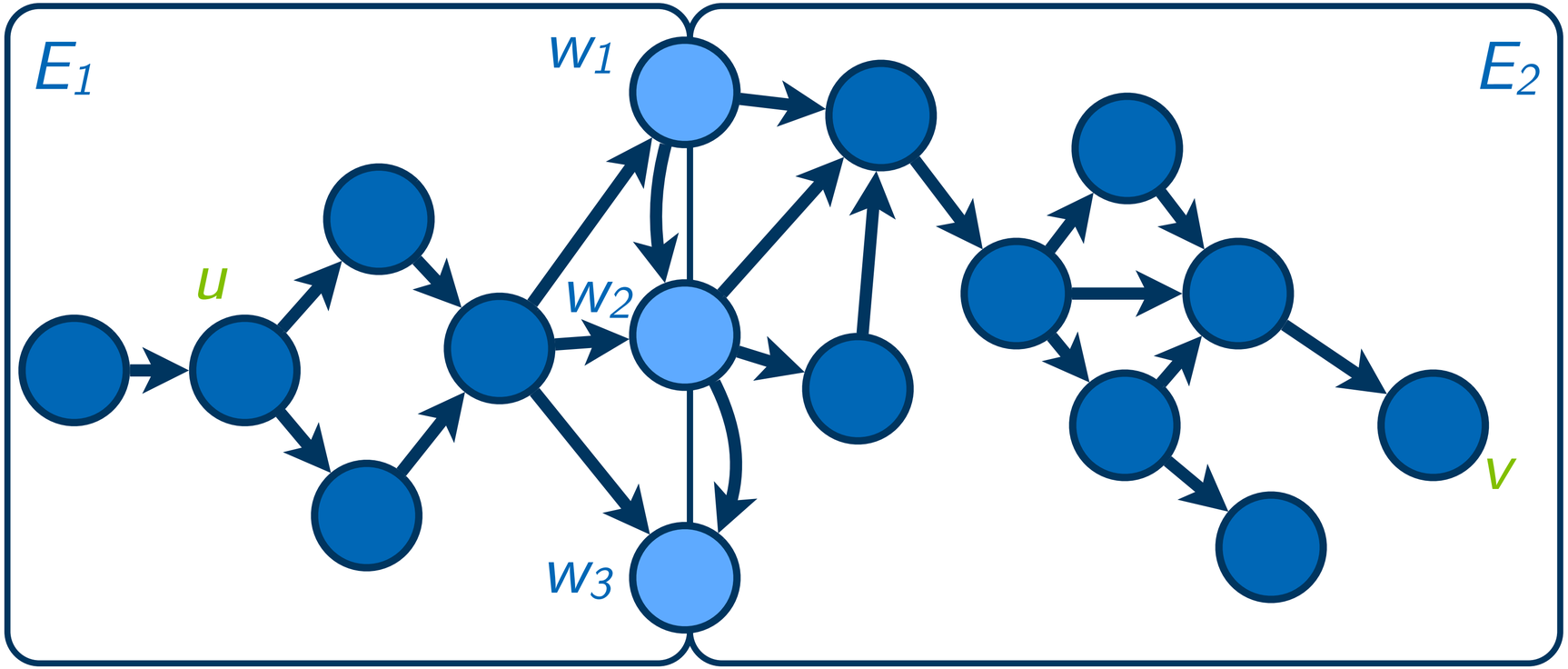

Extend DP-cases to: $Any[u, v] = \bigoplus_w (L[u, w] \otimes R[w, v])$

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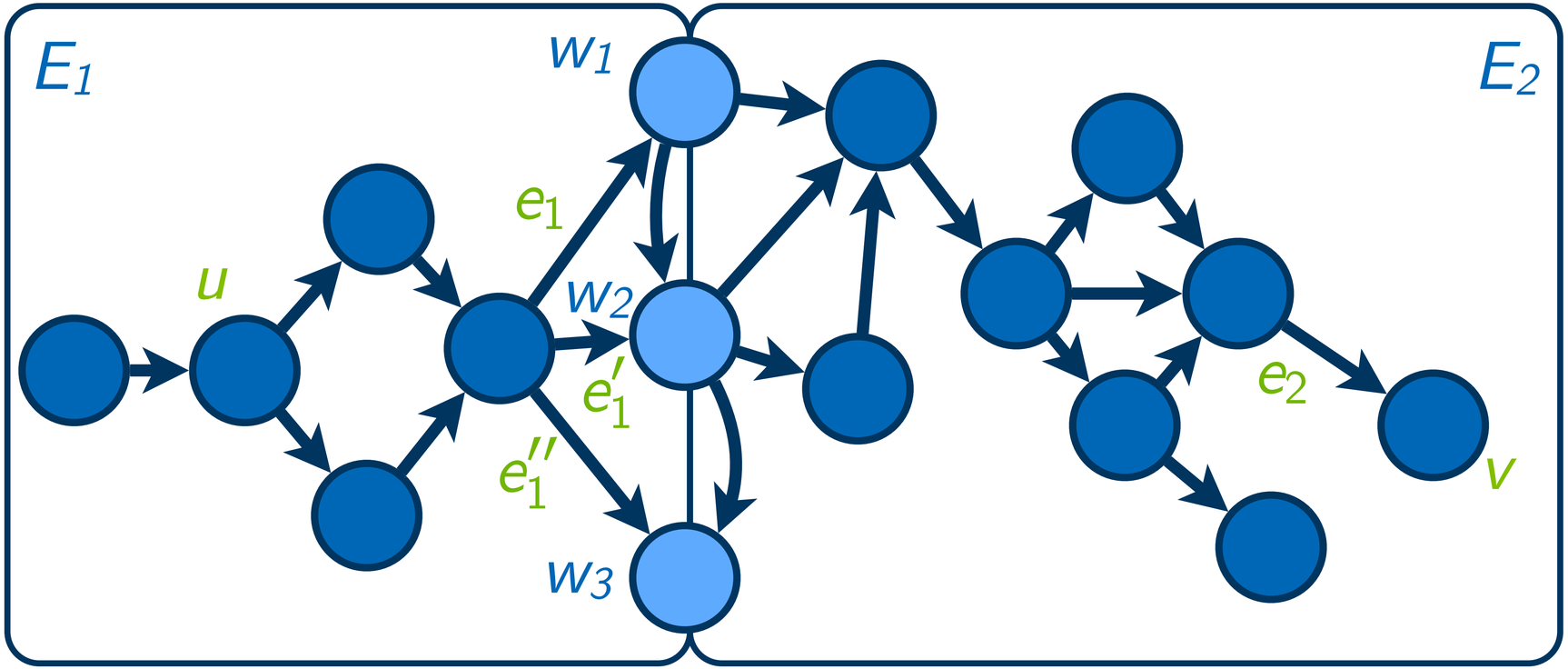

Example:

$$(e_1 \otimes e_2)$$
$$\oplus (e_1' \otimes e_2)$$
$$\oplus (e_1'' \otimes \bot)$$
$$= e_2 \oplus e_2 \oplus \bot$$
$$= e_2$$

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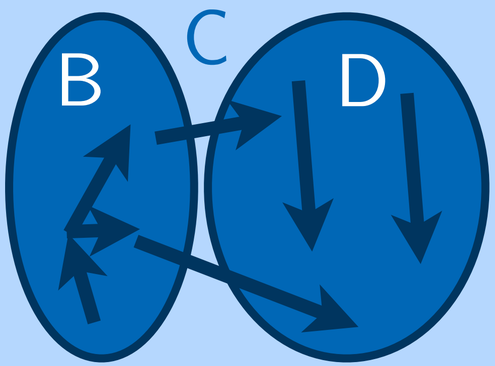
Recursion

vertex split



combine C, D

combine B, CD

Recursion

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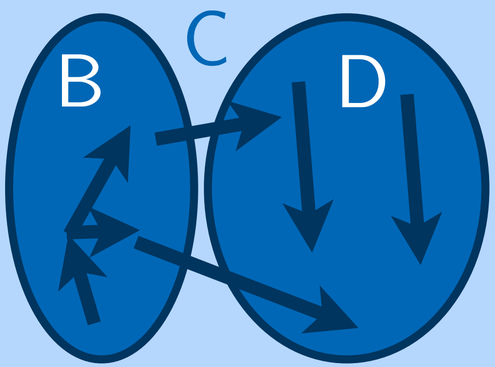

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Path Algebra

serial paths

$e \otimes \bot = \bot$

$e \otimes \top = e$

$e \otimes e' = e$ or e'

parallel paths

$\bot \oplus e = e$

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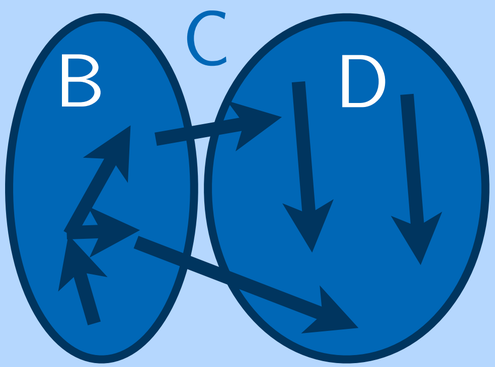

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$\{\bot, \top\} \cup E \to \{0,1\}^{\Theta(\log n)}$

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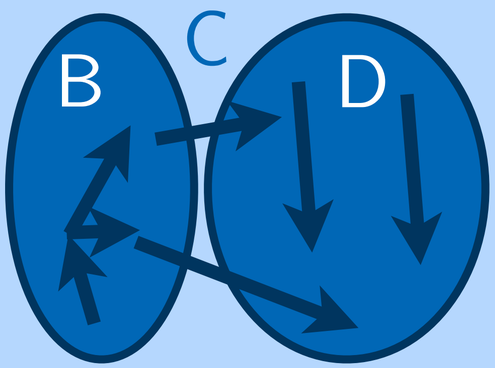

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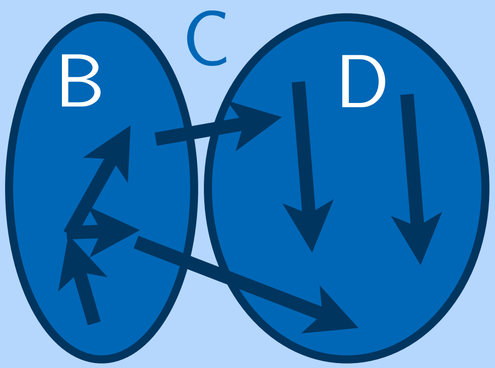

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DAGs / Divide & Conquer / Details / ETH zürich

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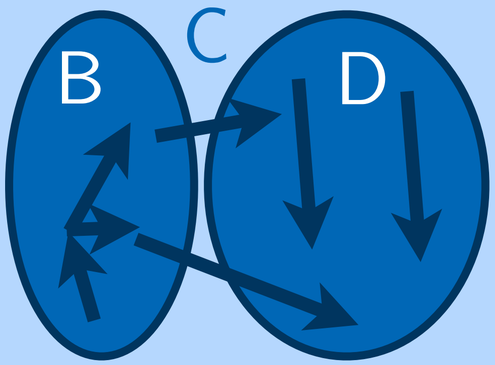

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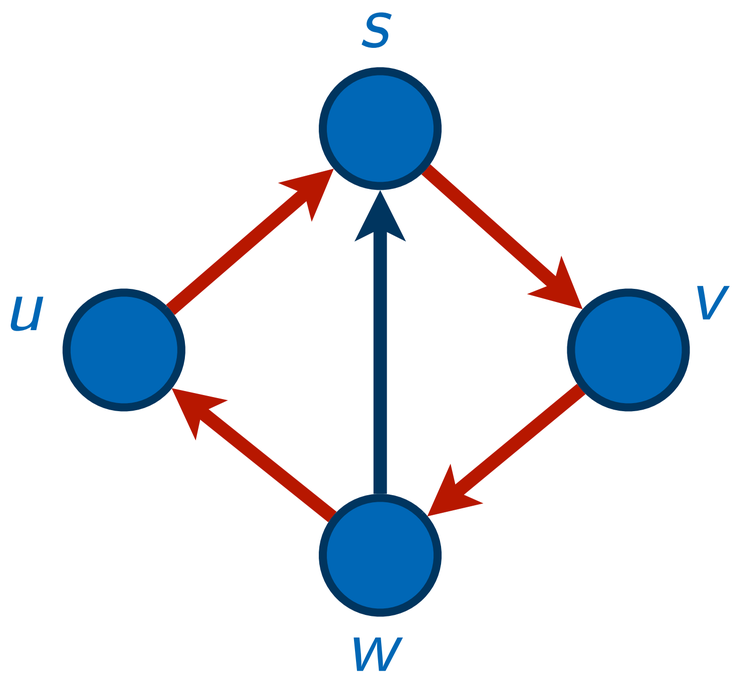
overall:
$\tilde{\mathcal{O}}(n^\omega)$

Strong bridge: edge whose removal increases number of SCCs



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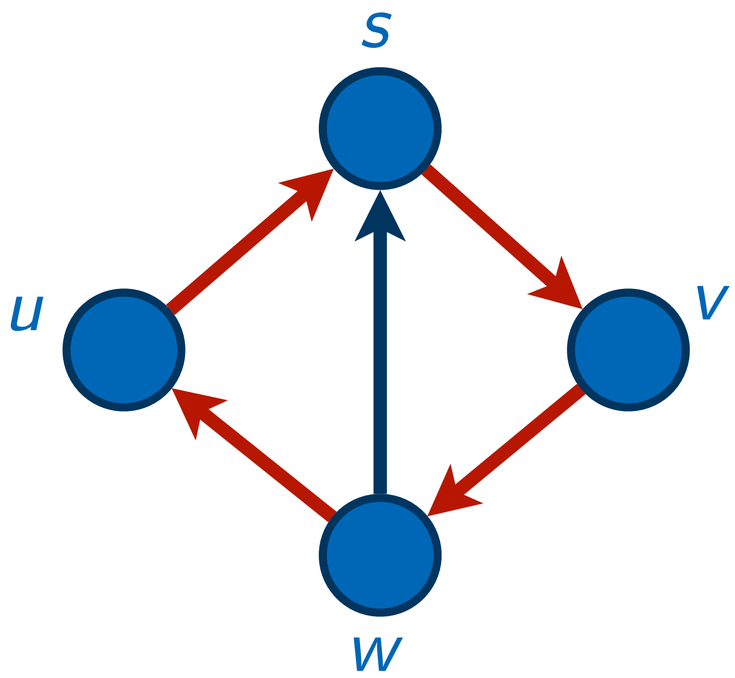

How to quickly find them all?

break the symmetry: „root“ G at s

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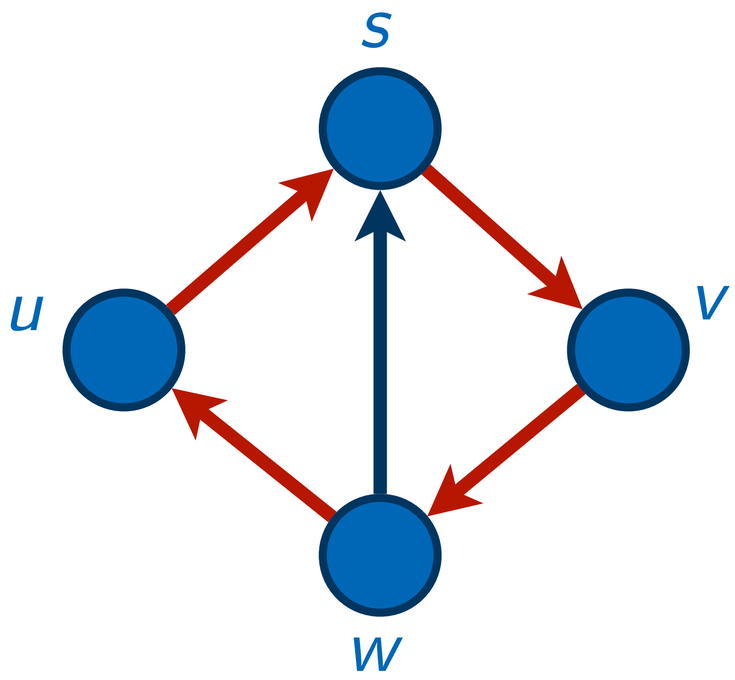

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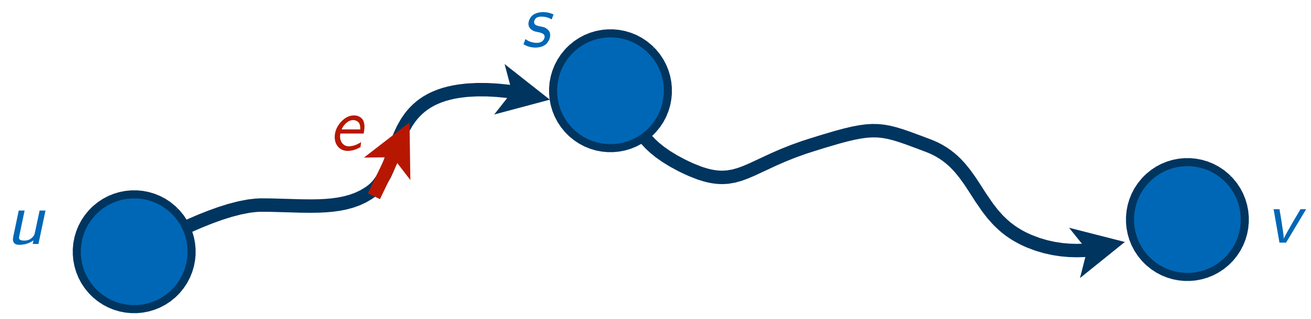
Observe: any strong bridge e disconnects some u from s or s from some v



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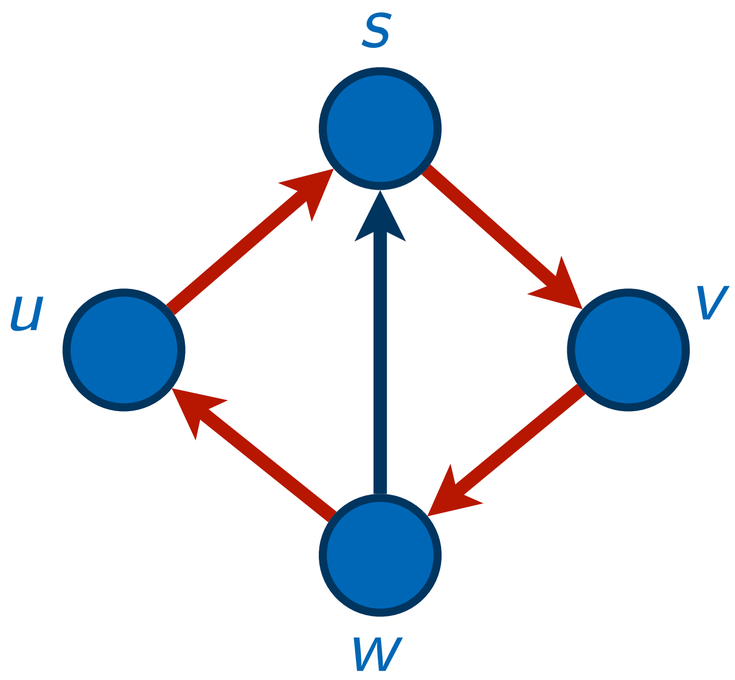

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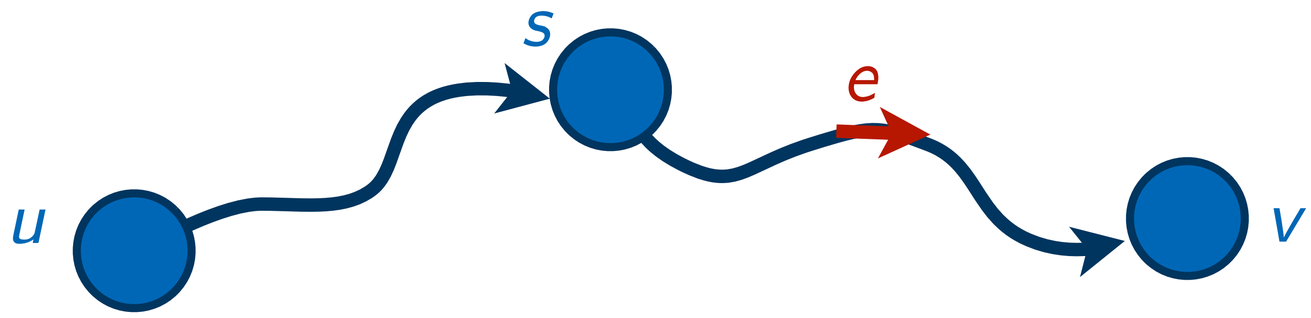
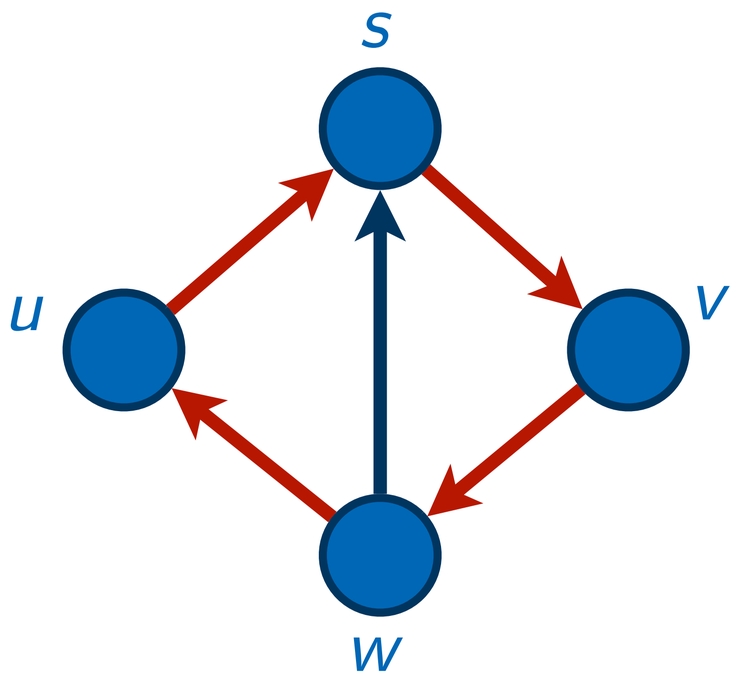


Strongly Connected / Strong Bridges

ETH zürich

Strong bridge: edge whose removal increases number of SCCs

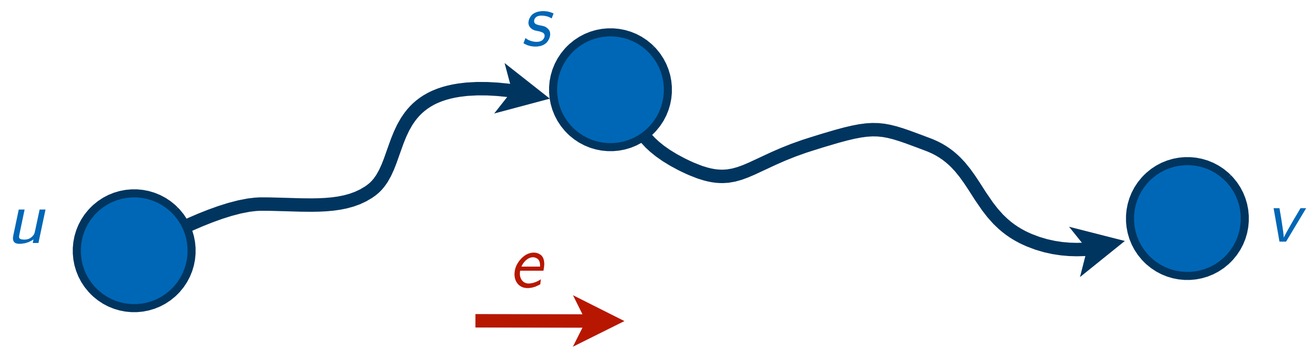


How to quickly find them all?

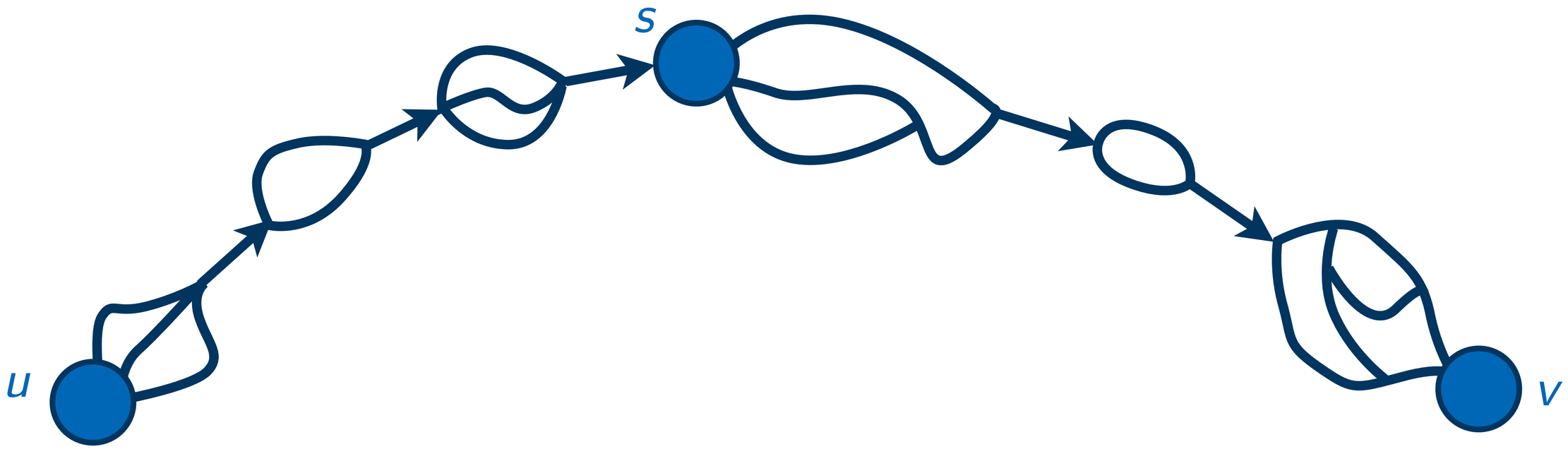
break the symmetry: „root“ G at s

Observe: any strong bridge e disconnects

some u from s or s from some v



Testing 2-reachability after breaking symmetry



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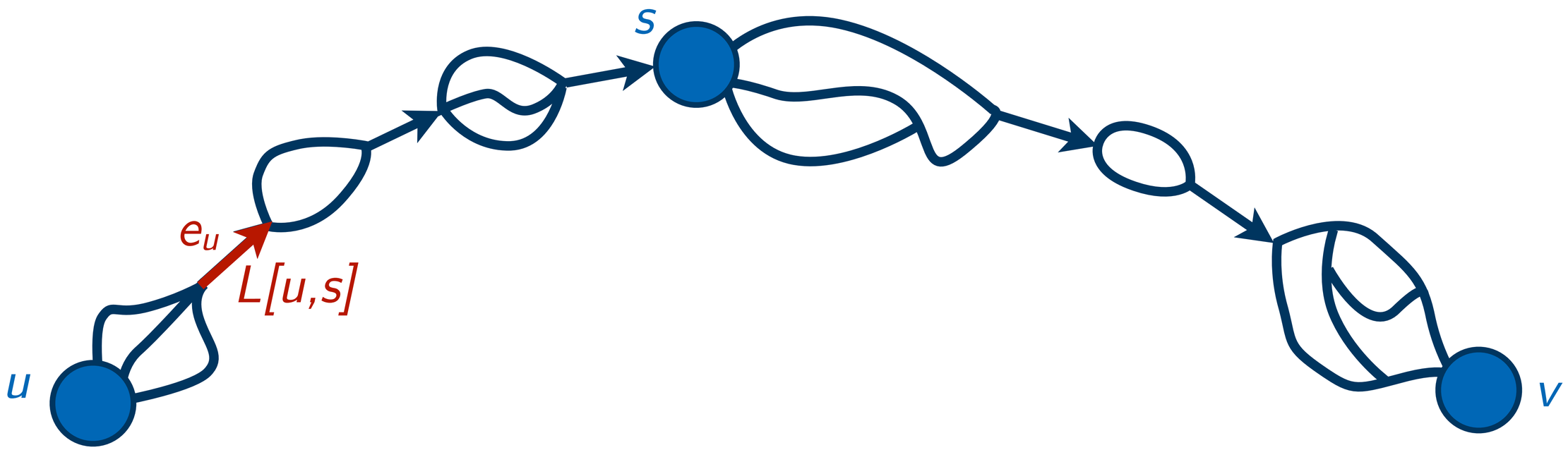

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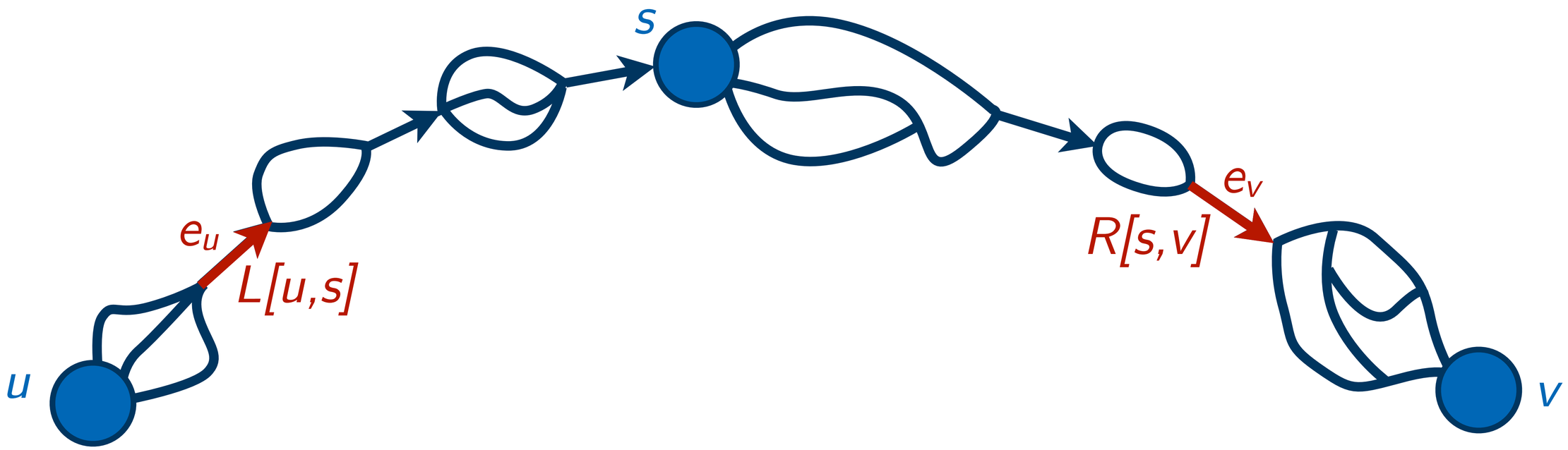

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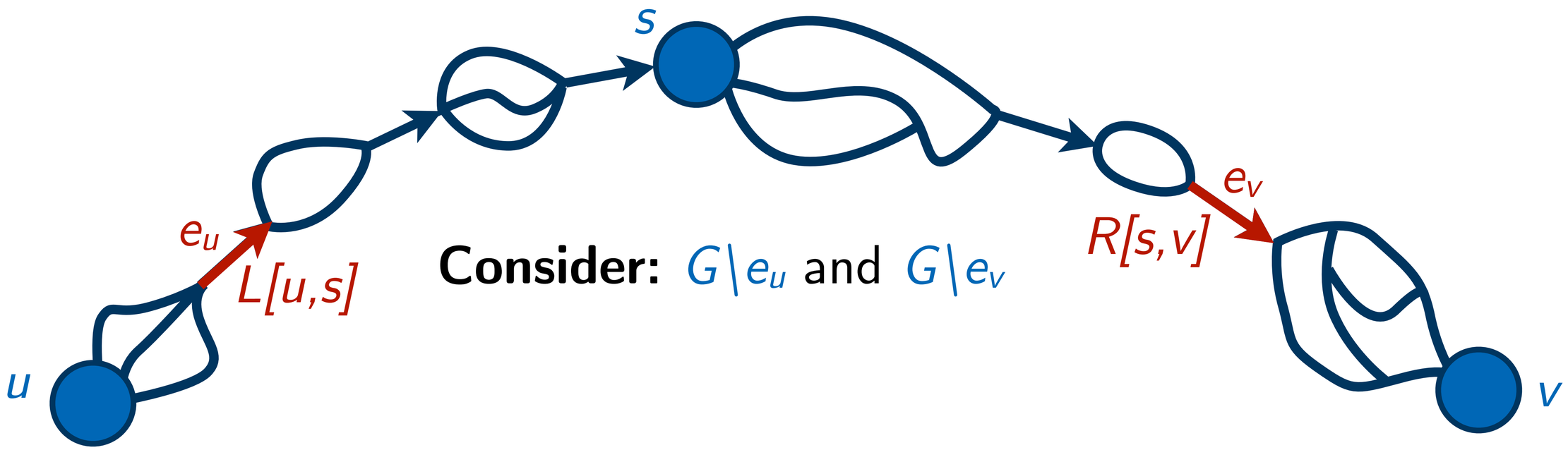

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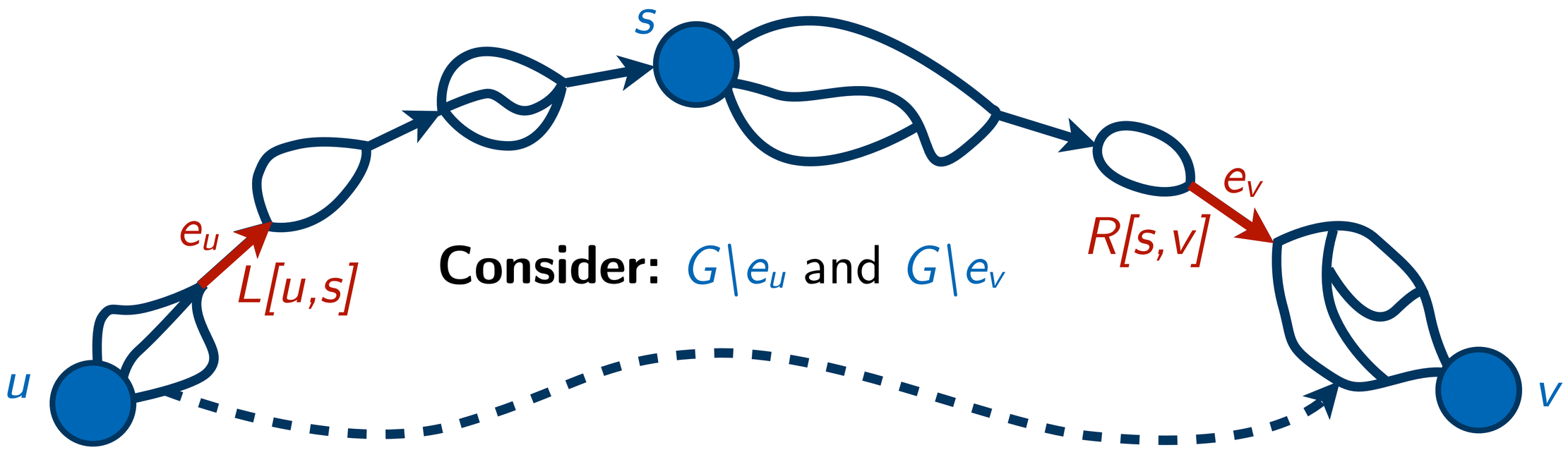

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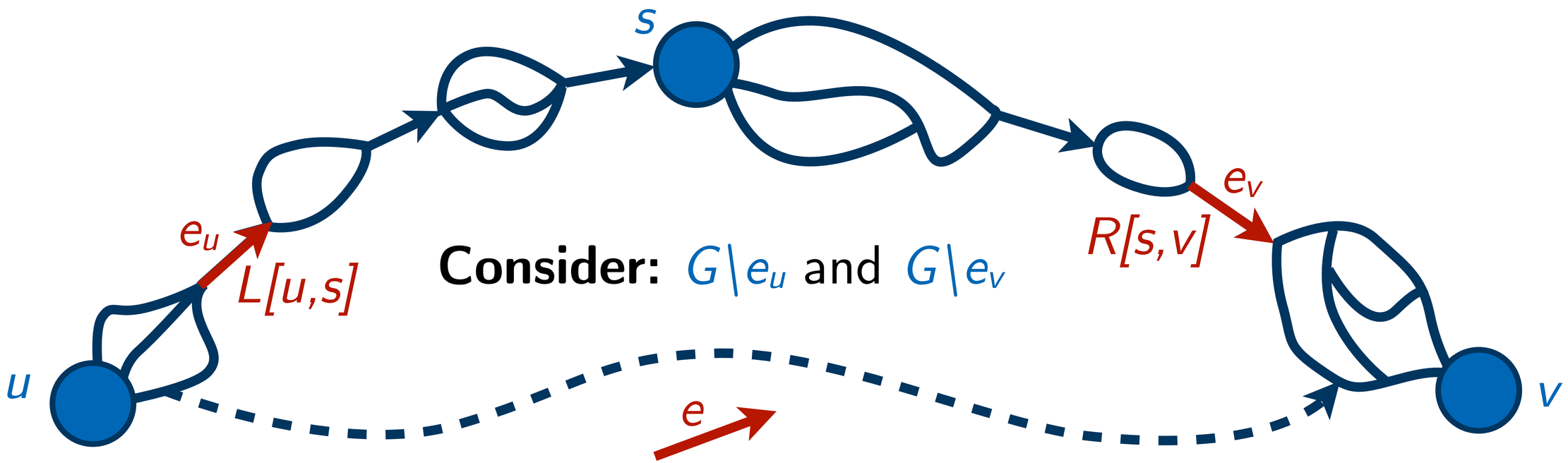


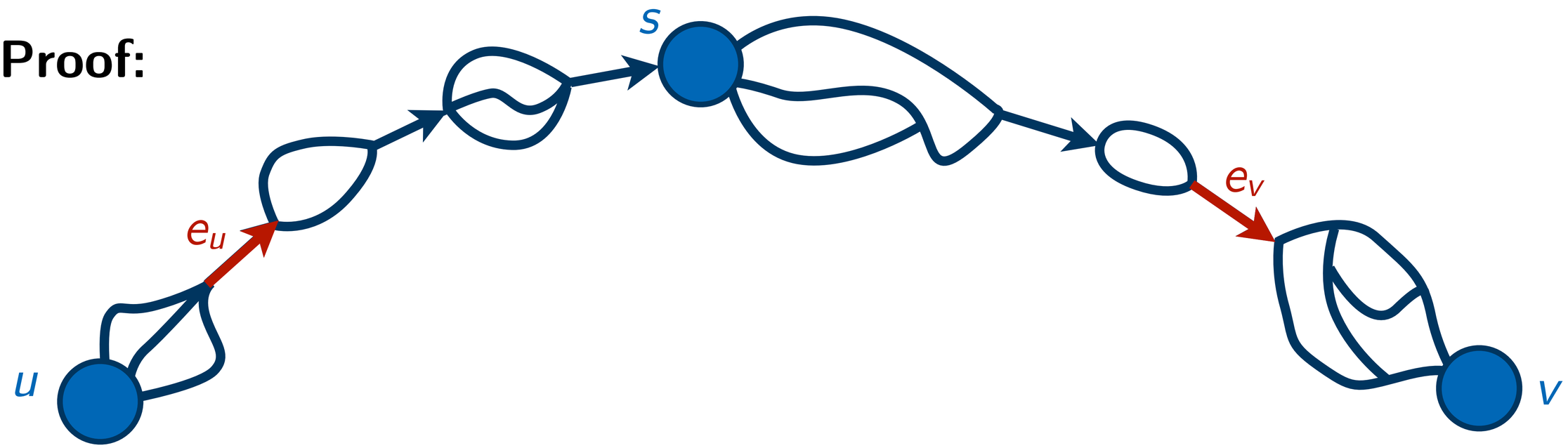
Claim	If some e disconnects u from v, then so does e_u or e_v.

Strongly Connected / Asymmetric

ETH zürich

Claim u-v-path in both $G \setminus e_u$ and $G \setminus e_v \Rightarrow$ multiple u-v-paths in G

Proof:



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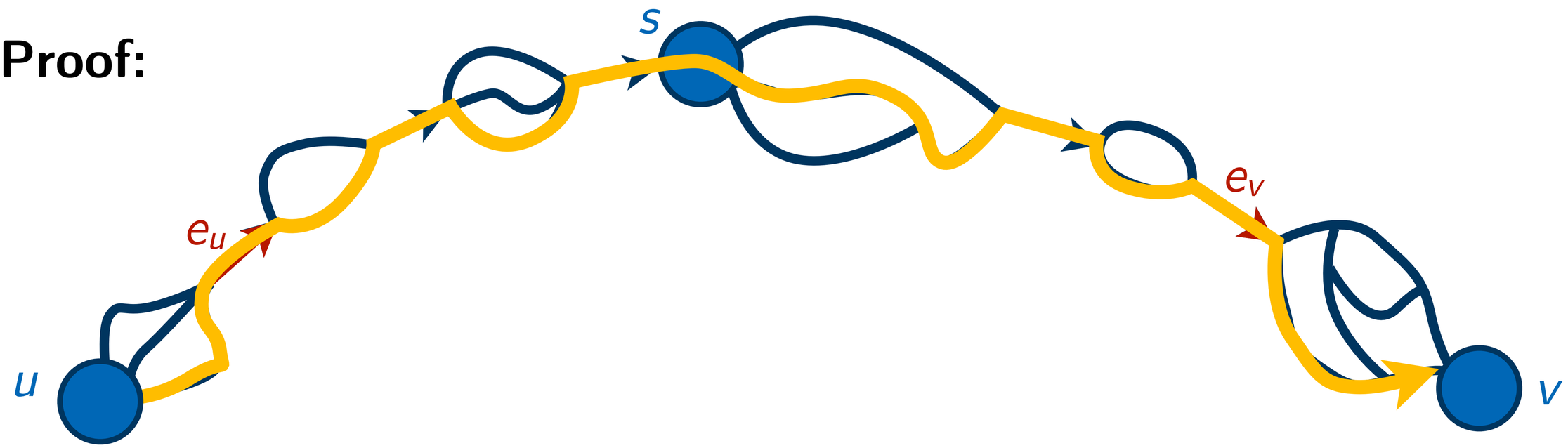

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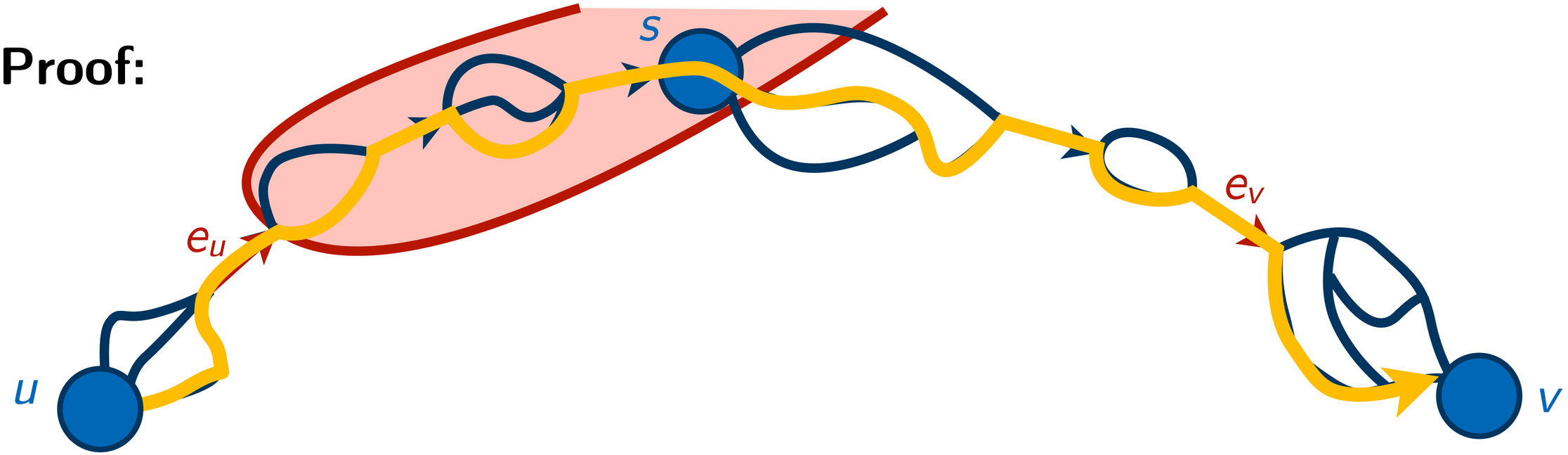

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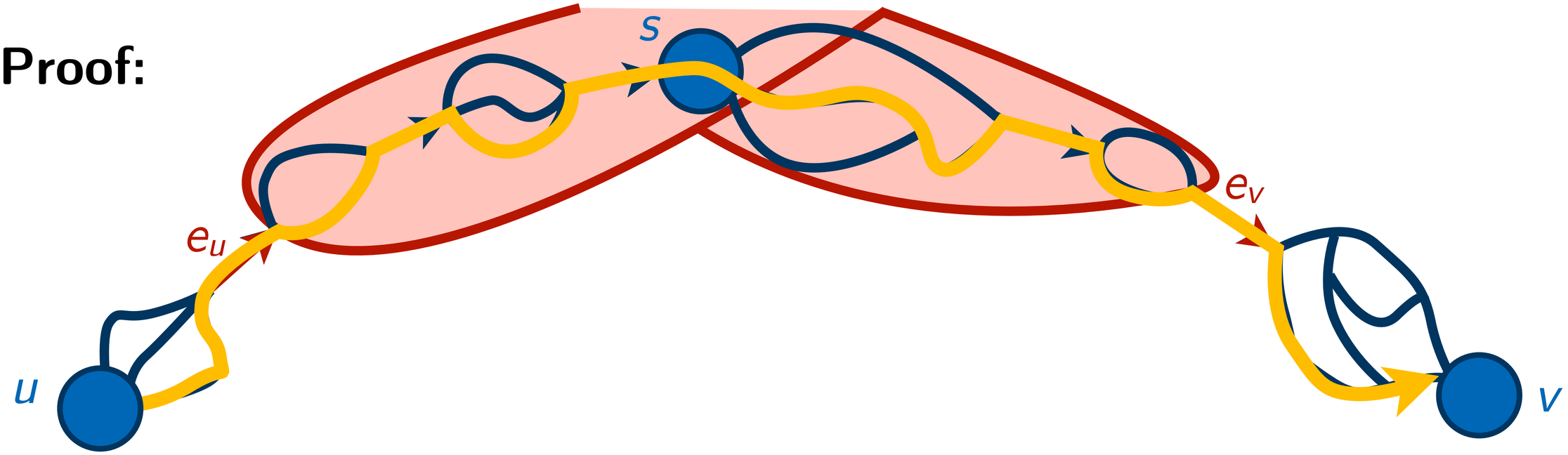

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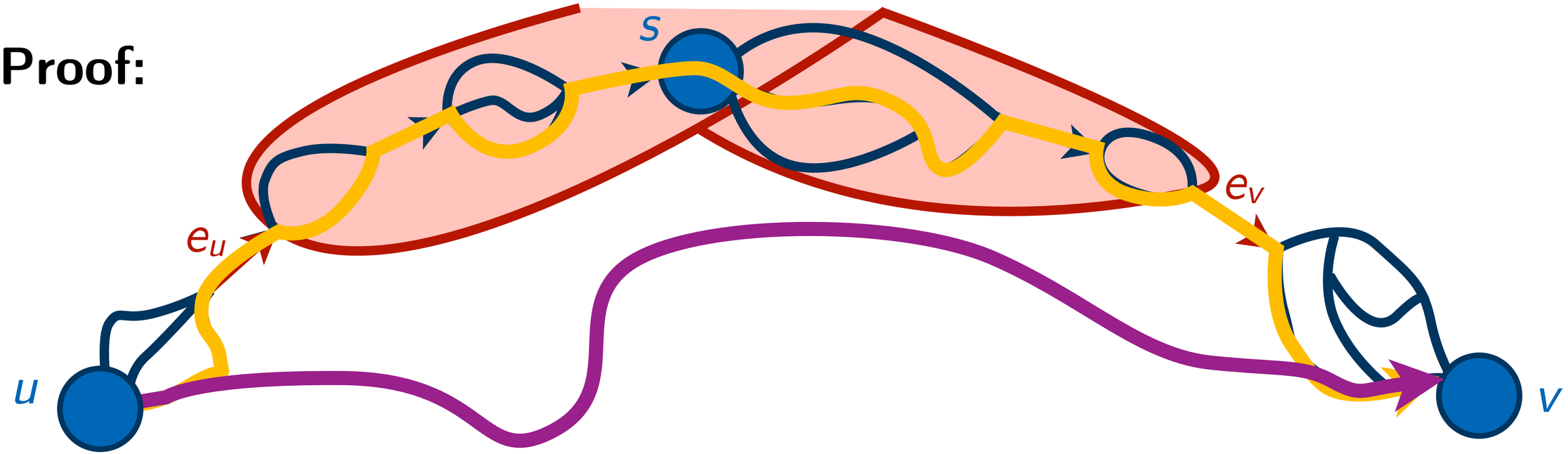

Case 1: path in $G \setminus \{e_u, e_v\} \Rightarrow$ 2 u-v-paths

Strongly Connected / Asymmetric

ETH zürich

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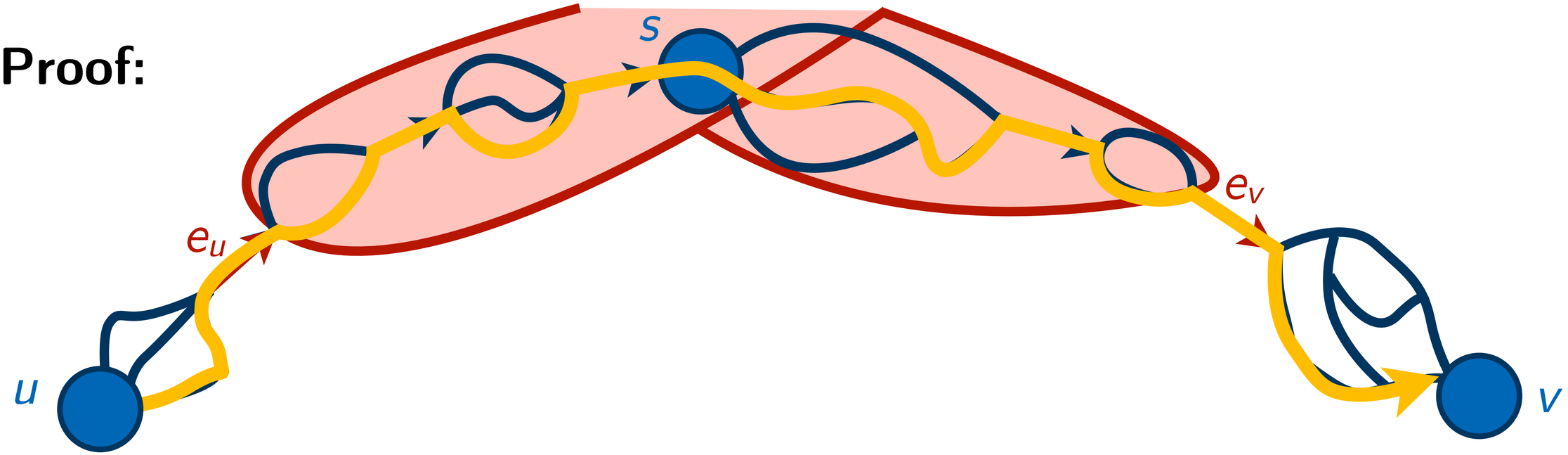

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Strongly Connected / Asymmetric

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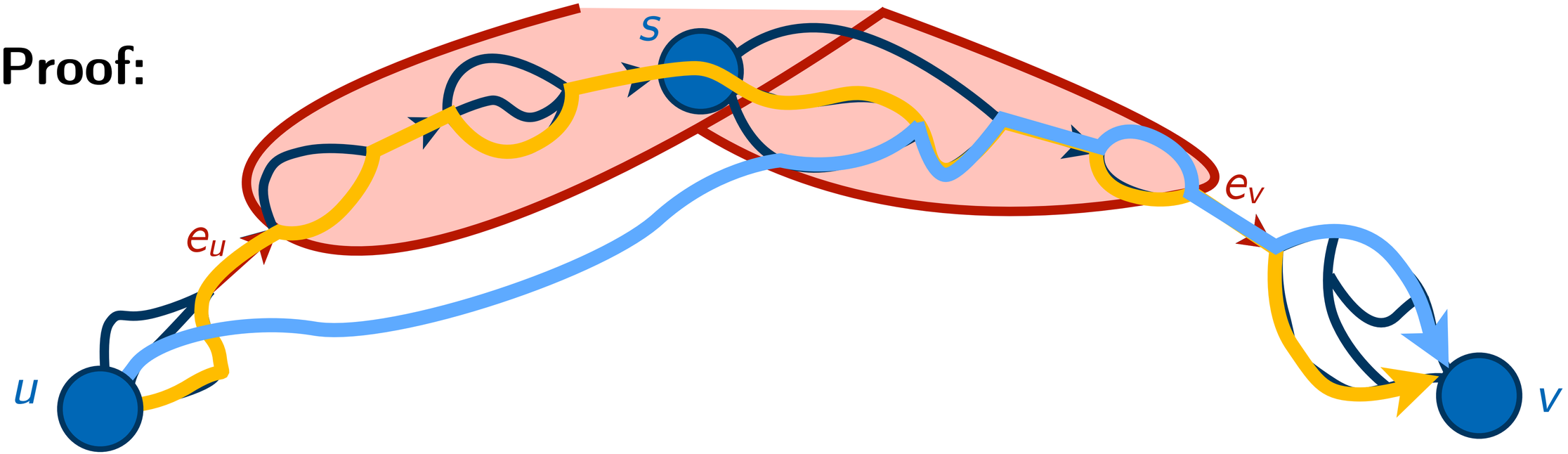

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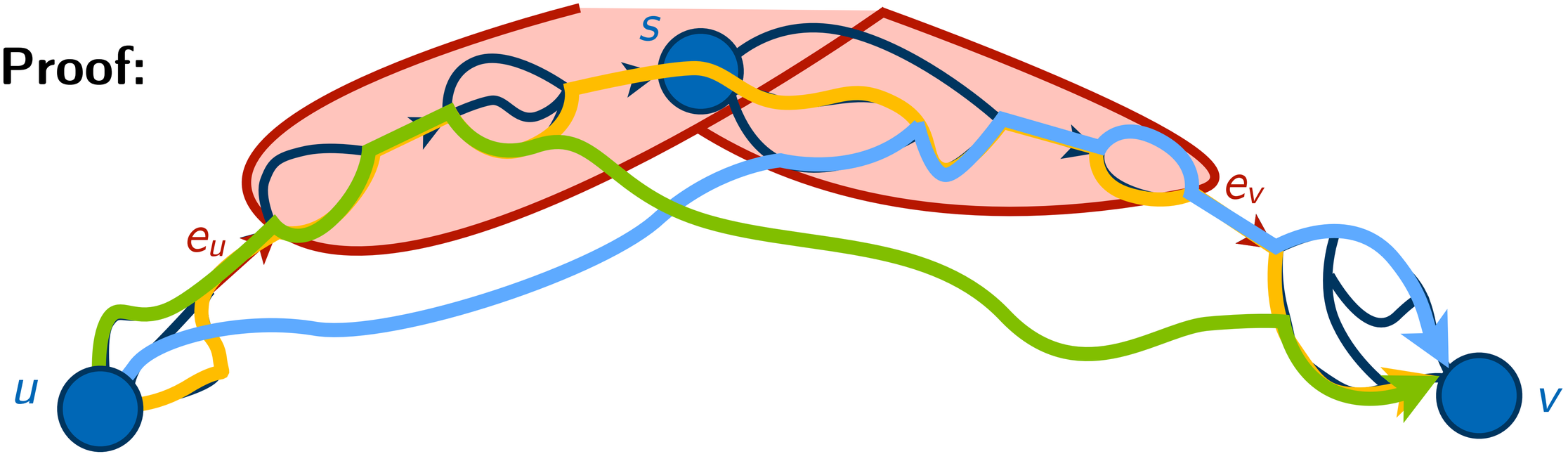

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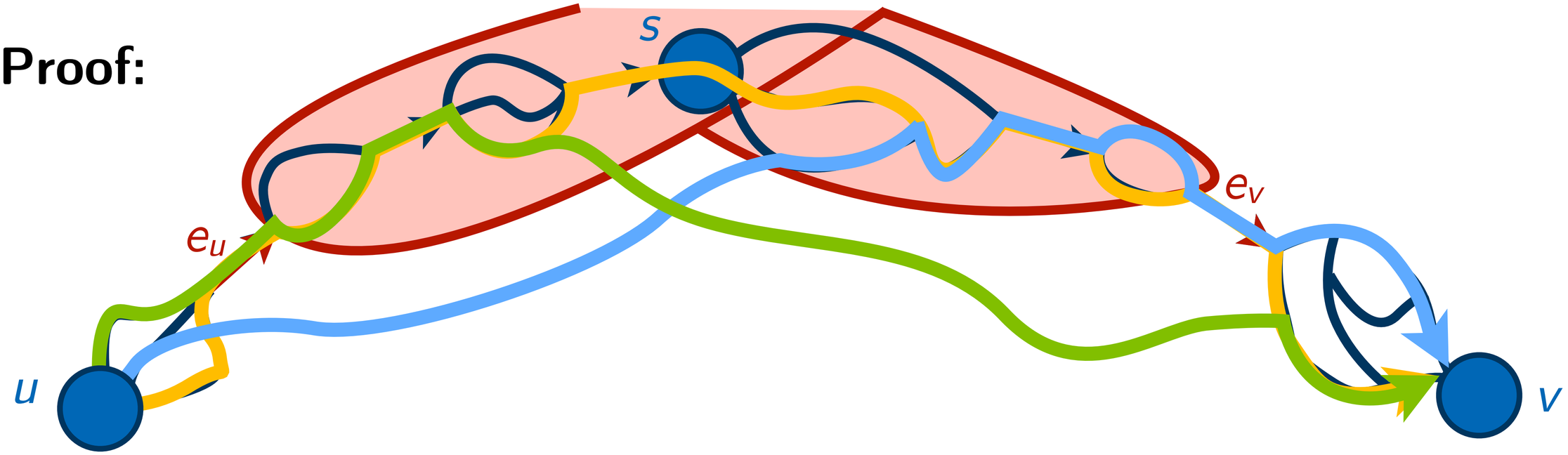

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Strongly Connected / Auxiliary Graphs / ETH zürich

Query

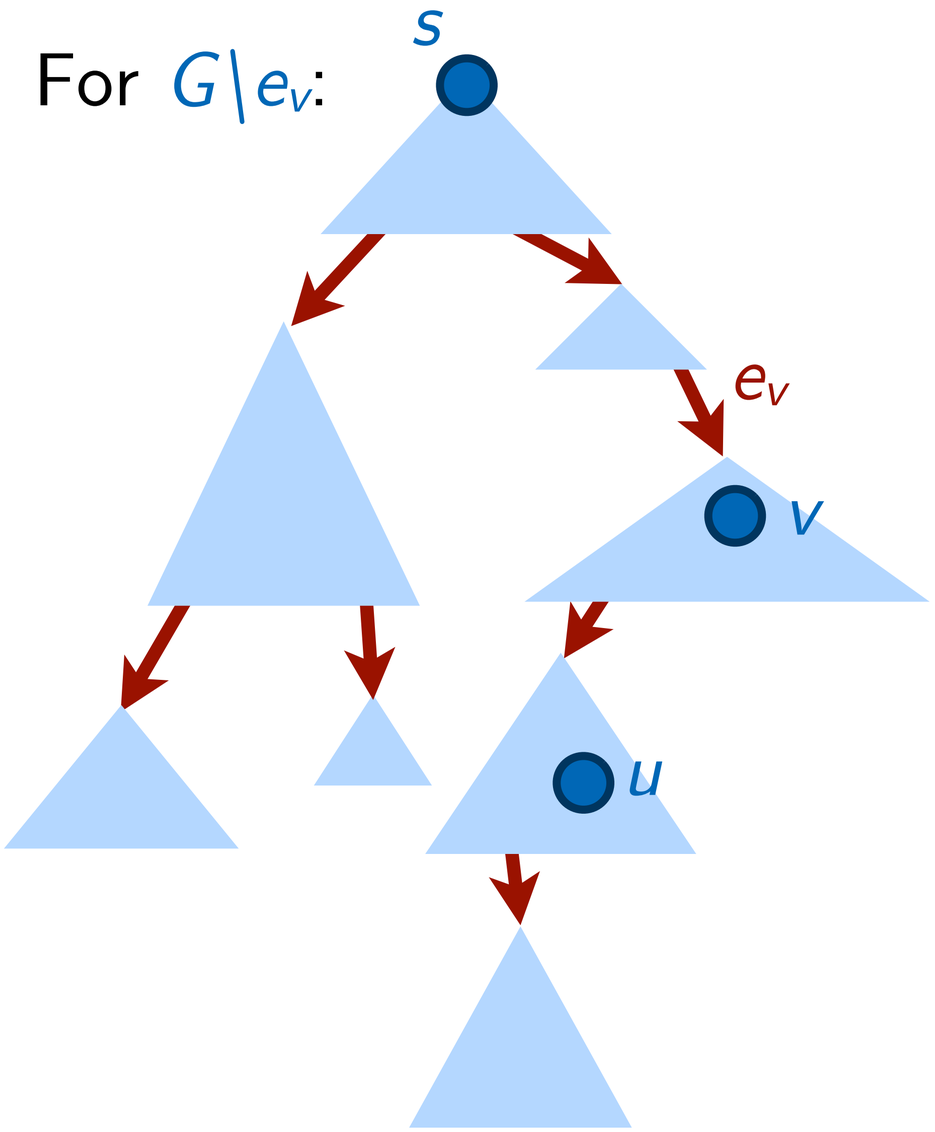
u-v-path in $G \setminus e_u$ and

u-v-path in $G \setminus e_v$?

Strongly Connected / Auxiliary Graphs

ETH zürich

Query	u-v-path in $G \setminus e_u$ and u-v-path in $G \setminus e_v$?



Strongly Connected / Auxiliary Graphs / ETH zürich

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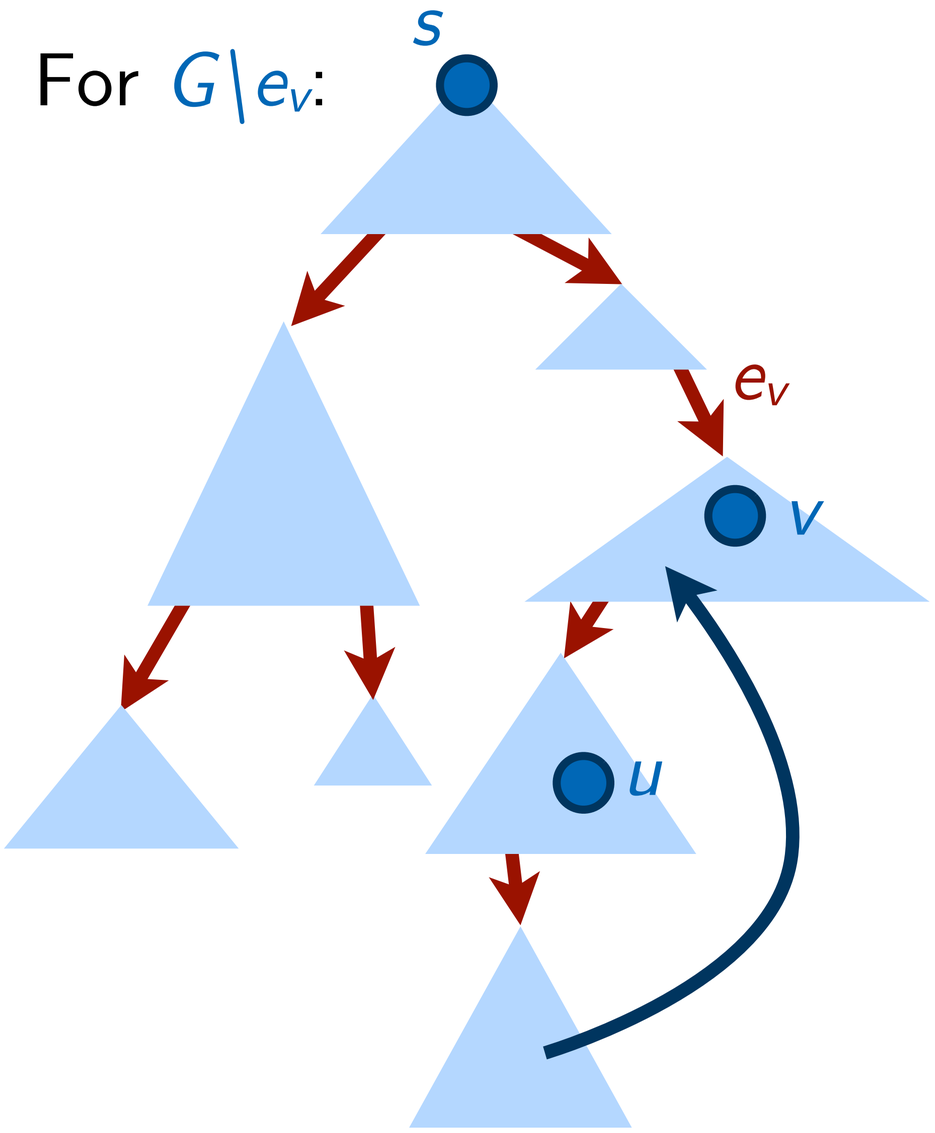

Strongly Connected / Auxiliary Graphs

ETH zürich

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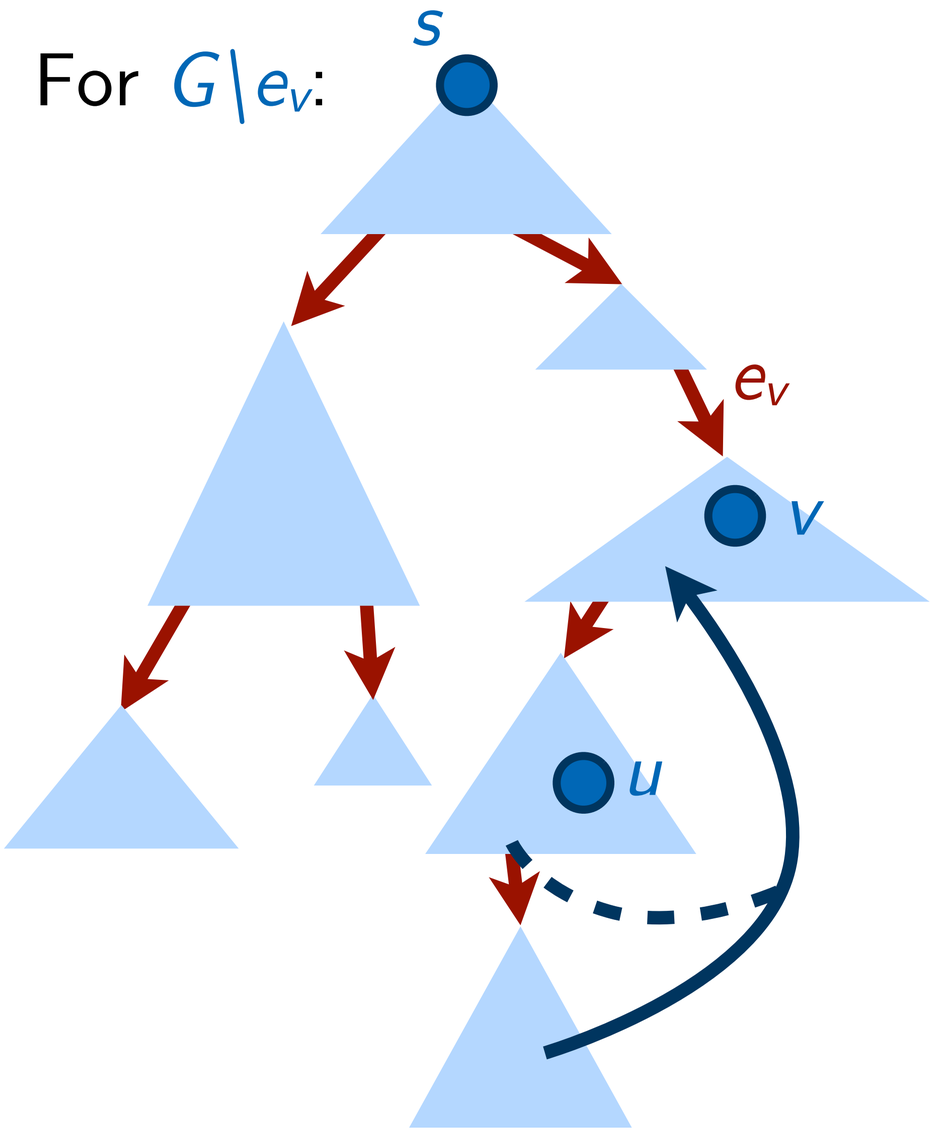

Strongly Connected / Auxiliary Graphs

ETH zürich

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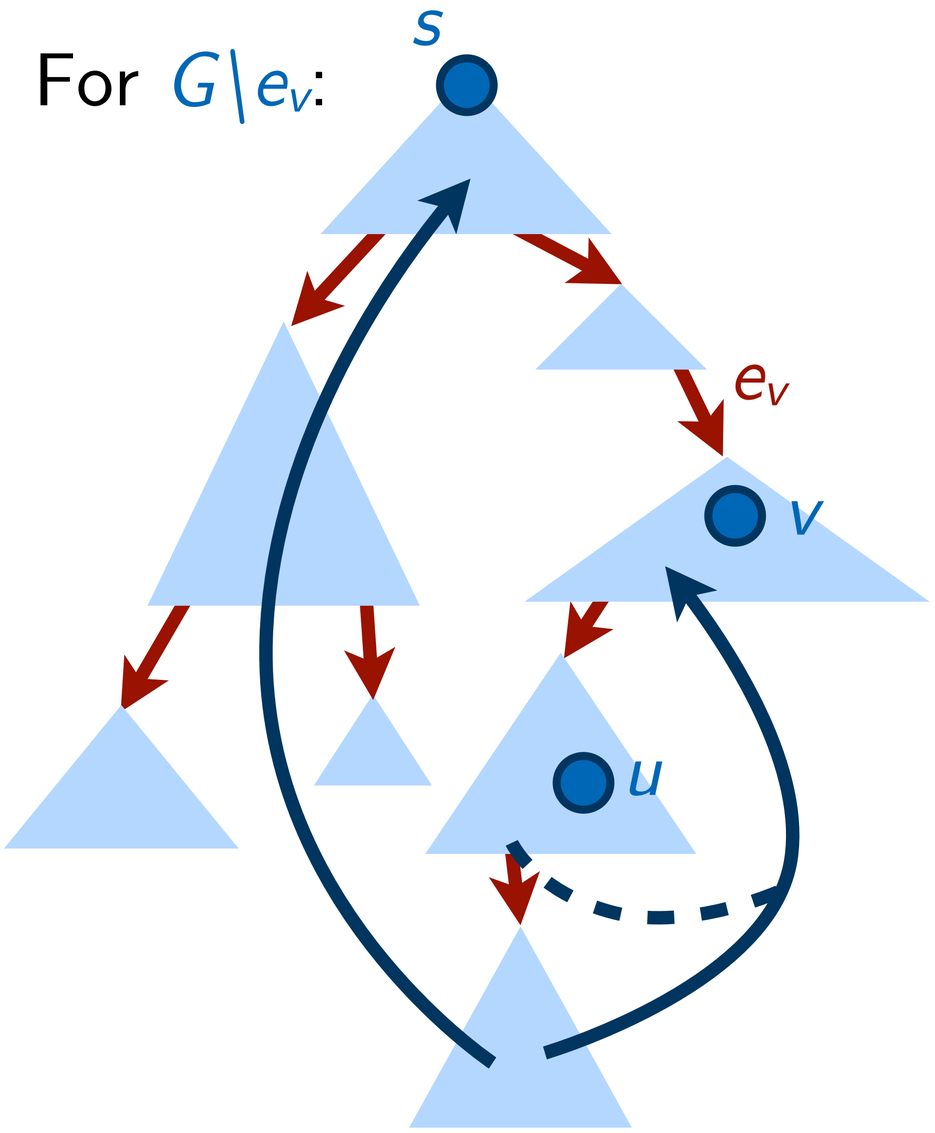

Strongly Connected Auxiliary Graphs

ETH zürich

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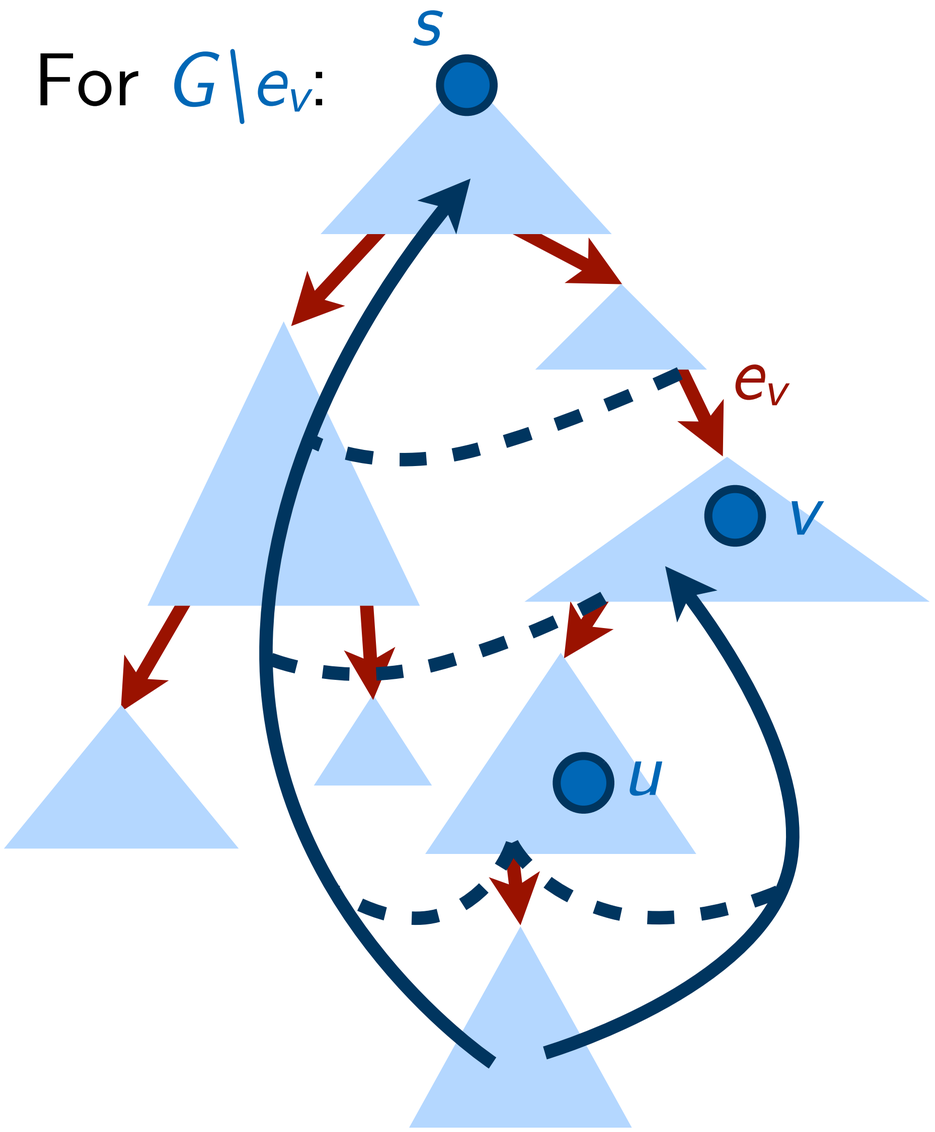

Strongly Connected Auxiliary Graphs

ETH zürich

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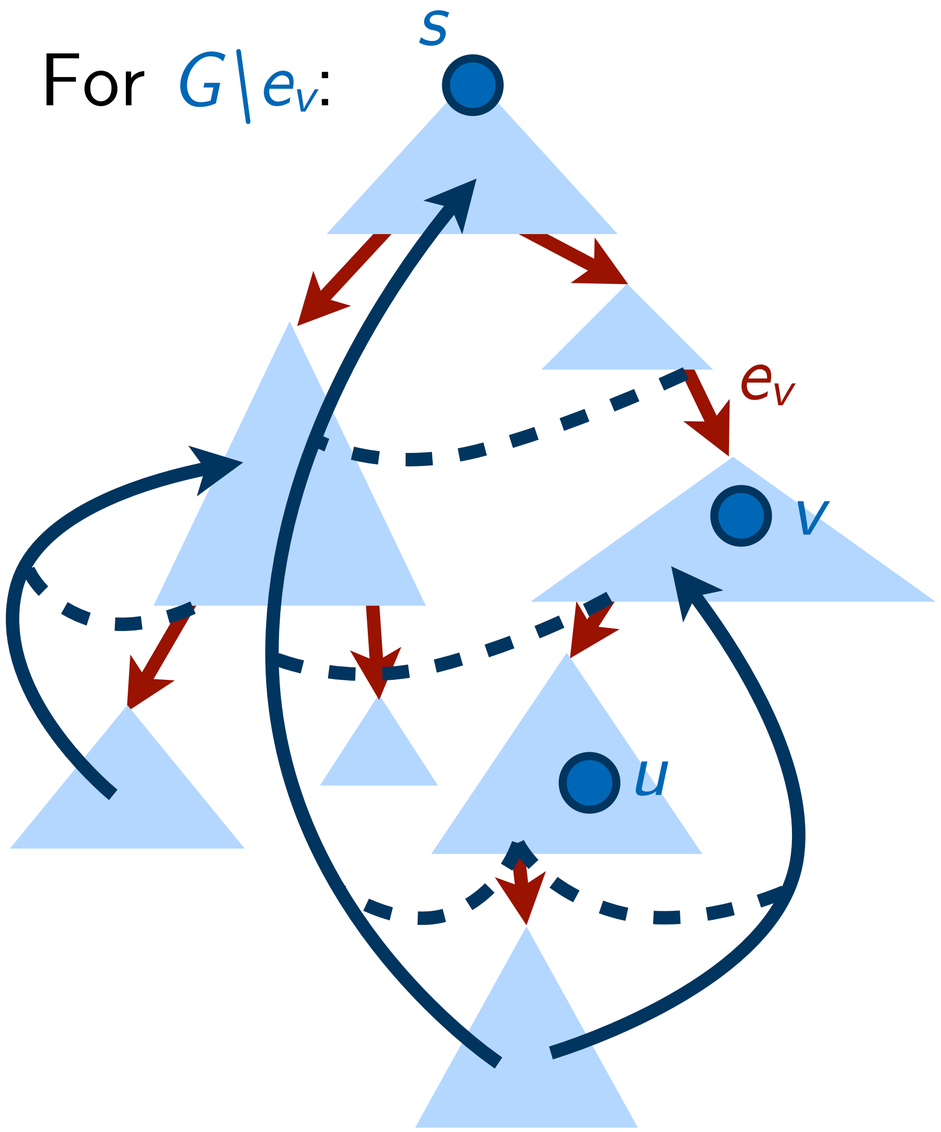
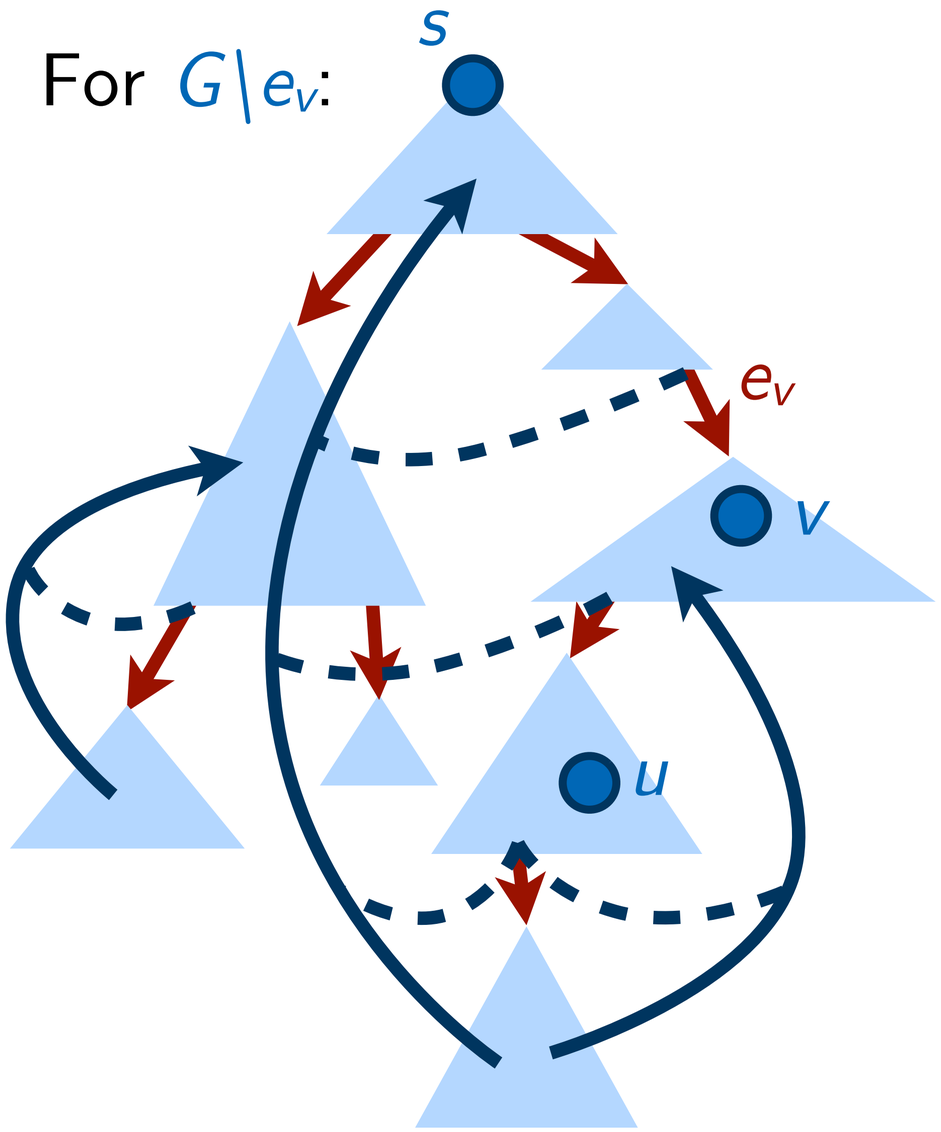


Strongly Connected / Auxiliary Graphs / ETH zürich

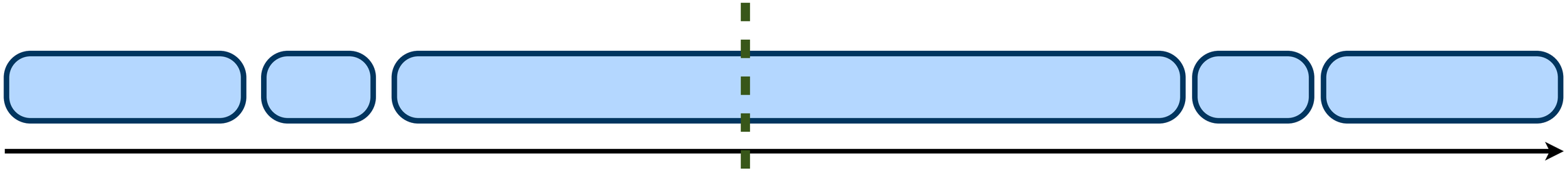
Query	u-v-path in $G \setminus e_u$ and u-v-path in $G \setminus e_v$?

Algorithm:

- build two auxiliary graphs in $\mathcal{O}(n^2)$
- take transitive closure in $\mathcal{O}(n^\omega)$
- answer queries in $\mathcal{O}(1)$



First: compute SCCs and any topological order in $\mathcal{O}(n+m)$



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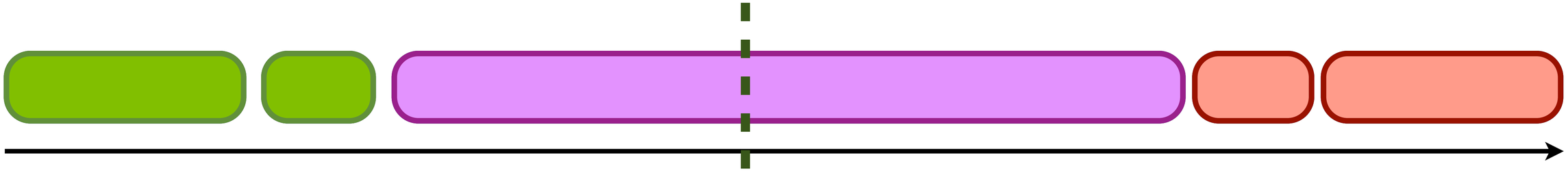
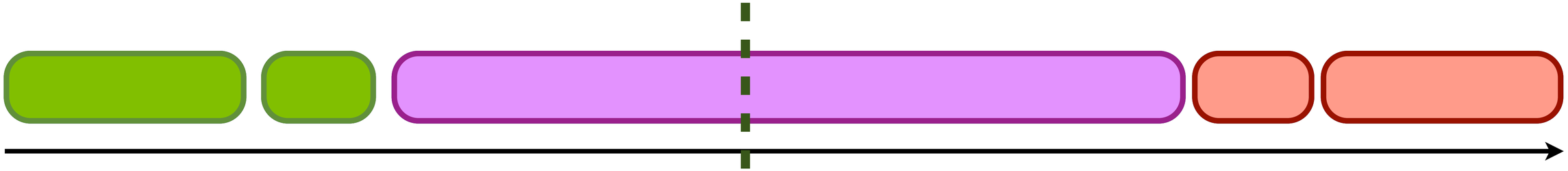


Is there a giant in the middle?

First: compute SCCs and any topological order in $\mathcal{O}(n+m)$



Is there a giant in the middle?

- Yes: $T(n) \leq T(n/3) + T(n/3) + \mathcal{O}(n^\omega) + \mathcal{O}(n^\omega \log n)$

recursive parts | giant SCC | merge edge-cuts

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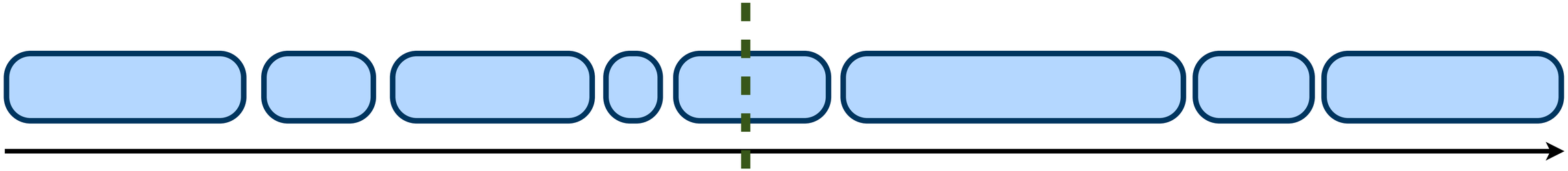

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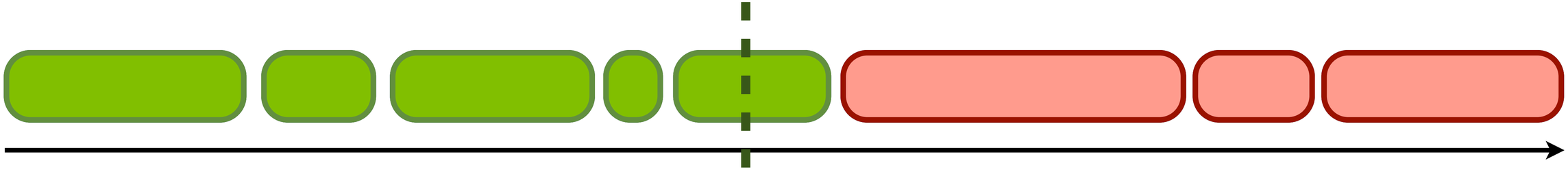

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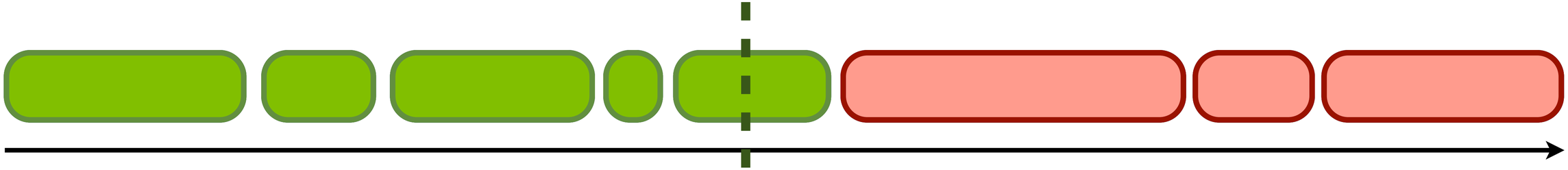

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recursive parts | giant SCC | merge edge-cuts

Dominating: $\mathcal{O}(n^\omega \log n)$

More 2-Reachability

ETH zürich

Main Result All-pairs 2-edge-reachability with witnesses in $\mathcal{O}(n^\omega \log n)$

More 2-Reachability

ETH zürich

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2-Reachability

- two edge-disjoint u-v-path?
- two vertex-disjoint u-v-path?
- u-v-path avoiding e?
- u-v-path avoiding w?

More 2-Reachability

ETH zürich

Main Result All-pairs 2-edge-reachability with witnesses in $\mathcal{O}(n^\omega \log n)$

2-Reachability

- two edge-disjoint u-v-path?
- two vertex-disjoint u-v-path?
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Preprocessing: $\mathcal{O}(n^\omega \log n)$

Query for arbitrary u,v,w,e: $\mathcal{O}(1)$

More 2-Reachability

ETH zürich

Main Result All-pairs 2-edge-reachability with witnesses in $\mathcal{O}(n^\omega \log n)$

2-Reachability

- two edge-disjoint u-v-path?
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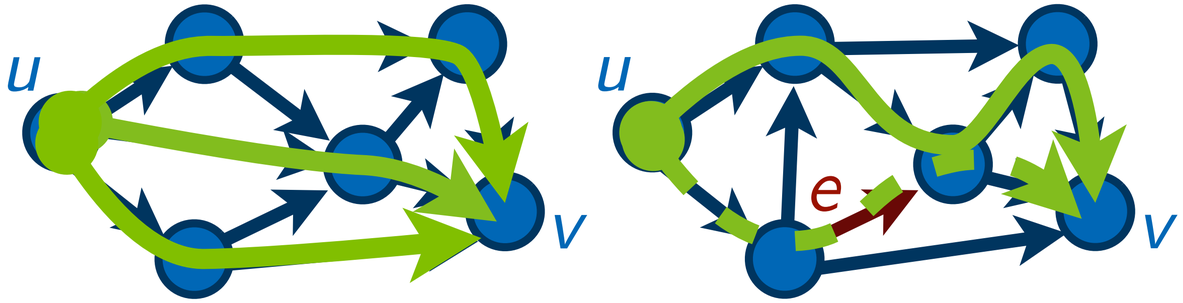
Open whether also in $\tilde{\mathcal{O}}(n^\omega)$

- k edge-disjoint u-v-path?
- replacement paths



Related work: fast digraph queries after preprocessing

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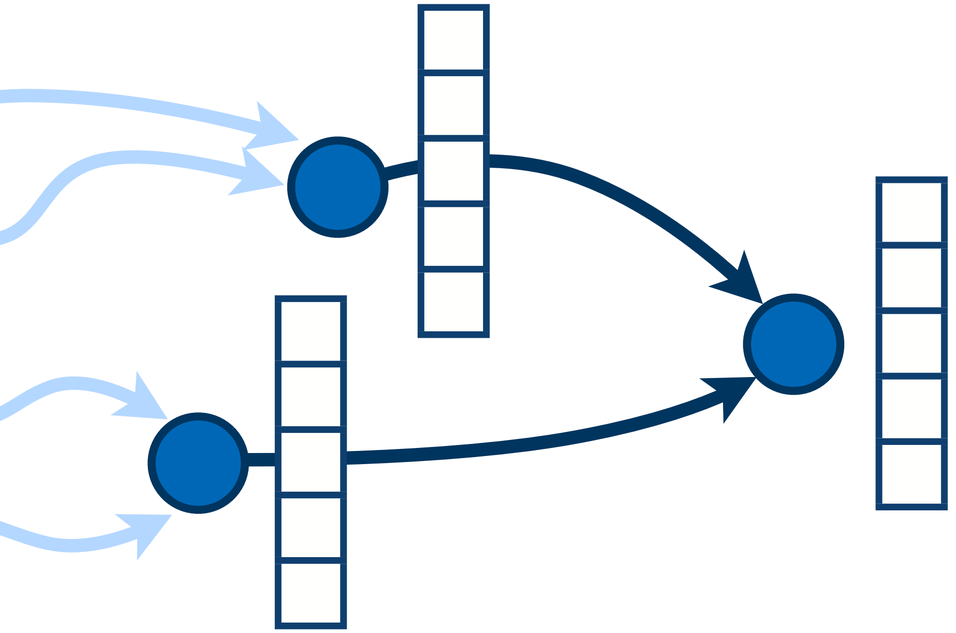
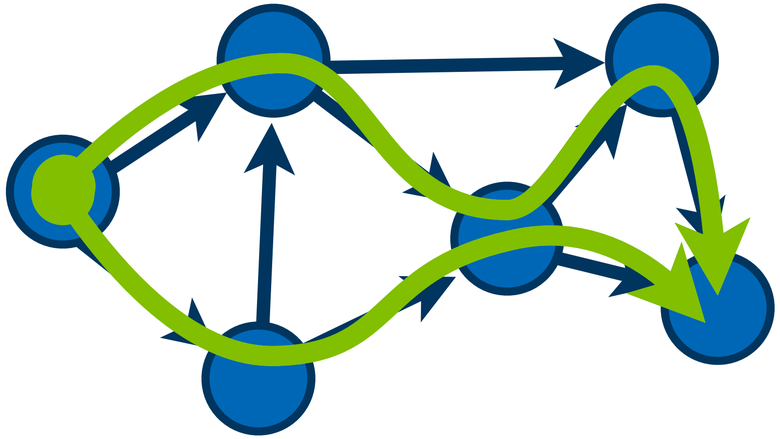
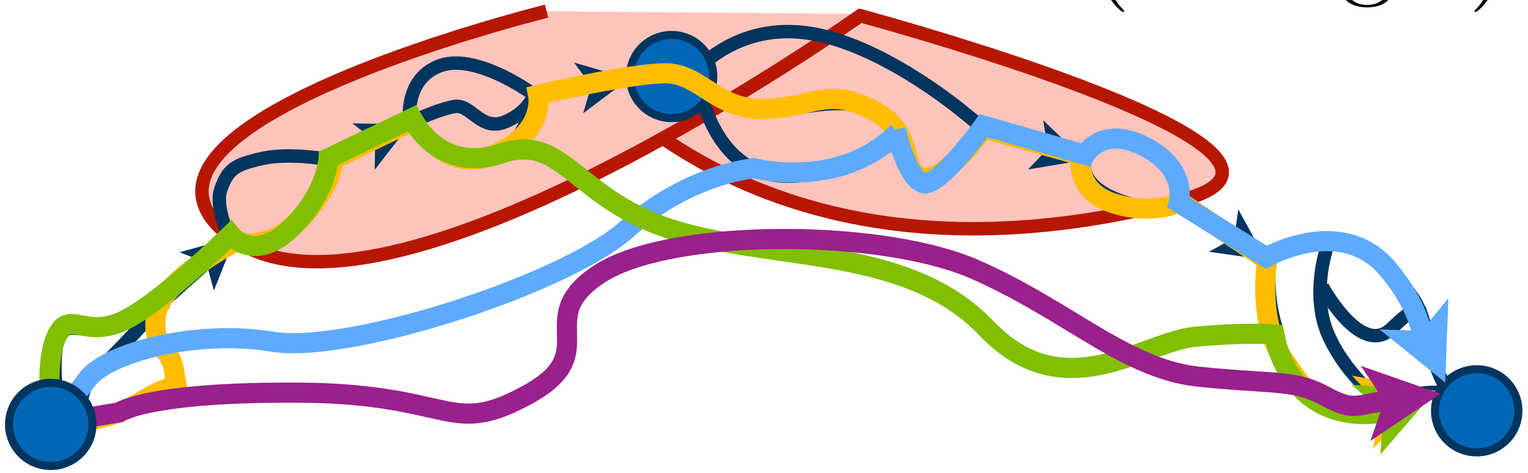
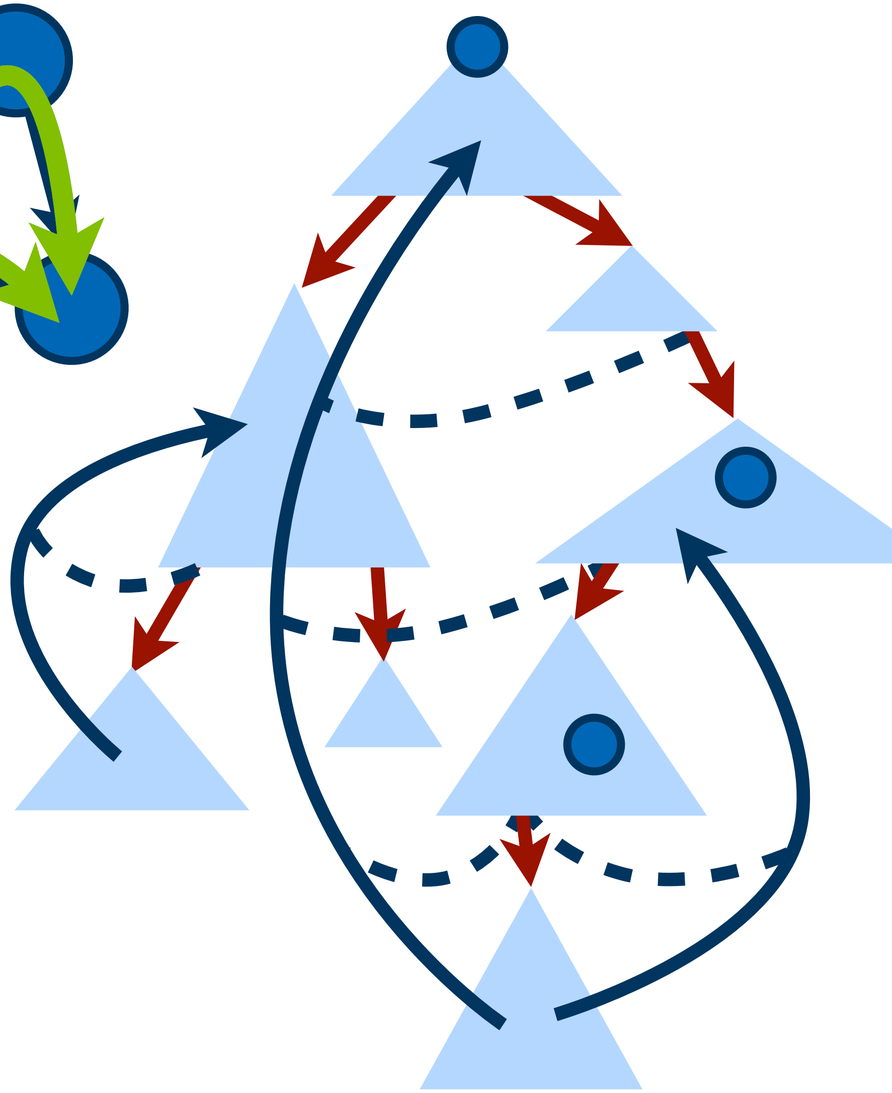
- Boolean mat. mult. with witnesses in $\mathcal{O}(n^\omega \log n)$ [Alon et al.'92]
- undirected shortest paths in $\mathcal{O}(n^\omega \log n)$ [Seidel'92]
- directed shortest paths in $\mathcal{O}(n^{(3+\omega)/2} \log n)$ [Alon et al.'92]
- bottleneck paths in $\mathcal{O}(n^{(3+\omega)/2} \log n)$ [Duan, Pettie'09]
- s-t-replacement shortest paths in $\tilde{\mathcal{O}}(n^\omega)$ [Williams'10]
- common ancestor in $\mathcal{O}(n^\omega)$ [Bender et al.'05]
- lowest common ancestor in $\mathcal{O}(n^{2+1/(4-\omega)})$ [Kowaluk, Lingas'05]
- junction in $\tilde{\mathcal{O}}(n^\omega)$ [Yuster'08]

Thank you for your attention!

ETH zürich





$$\mathcal{O}(n^{\omega} \log n)$$