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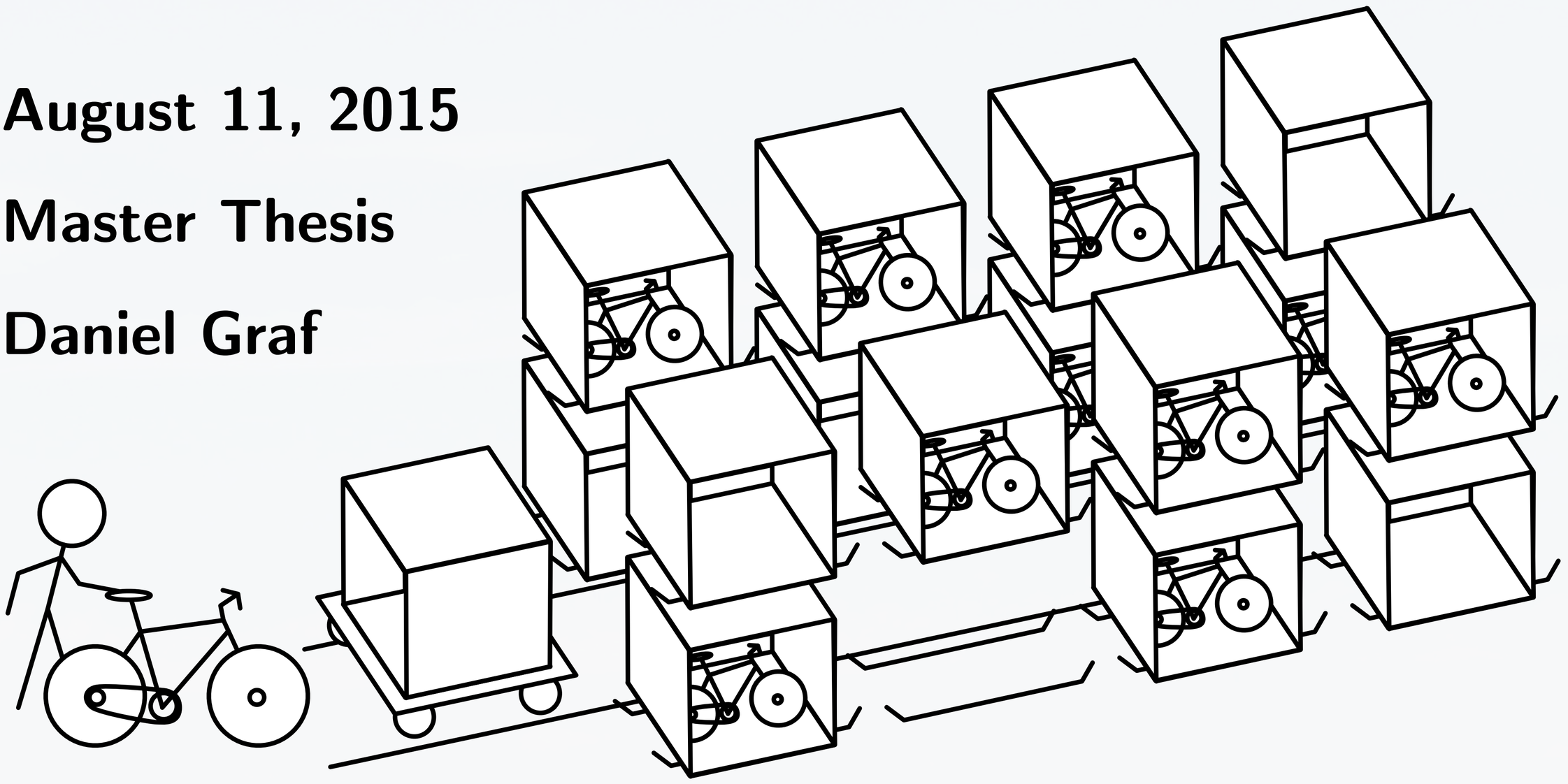
# Scheduling and Sorting Algorithms for Robotic Warehousing Systems

**August 11, 2015**

**Master Thesis**

**Daniel Graf**



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# Motivation



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# Bike Loft

1. Introduction
1.1 Motivation


Video by 3sat and Bike Loft GmbH

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# Bike Loft

1. Introduction
1.1 Motivation

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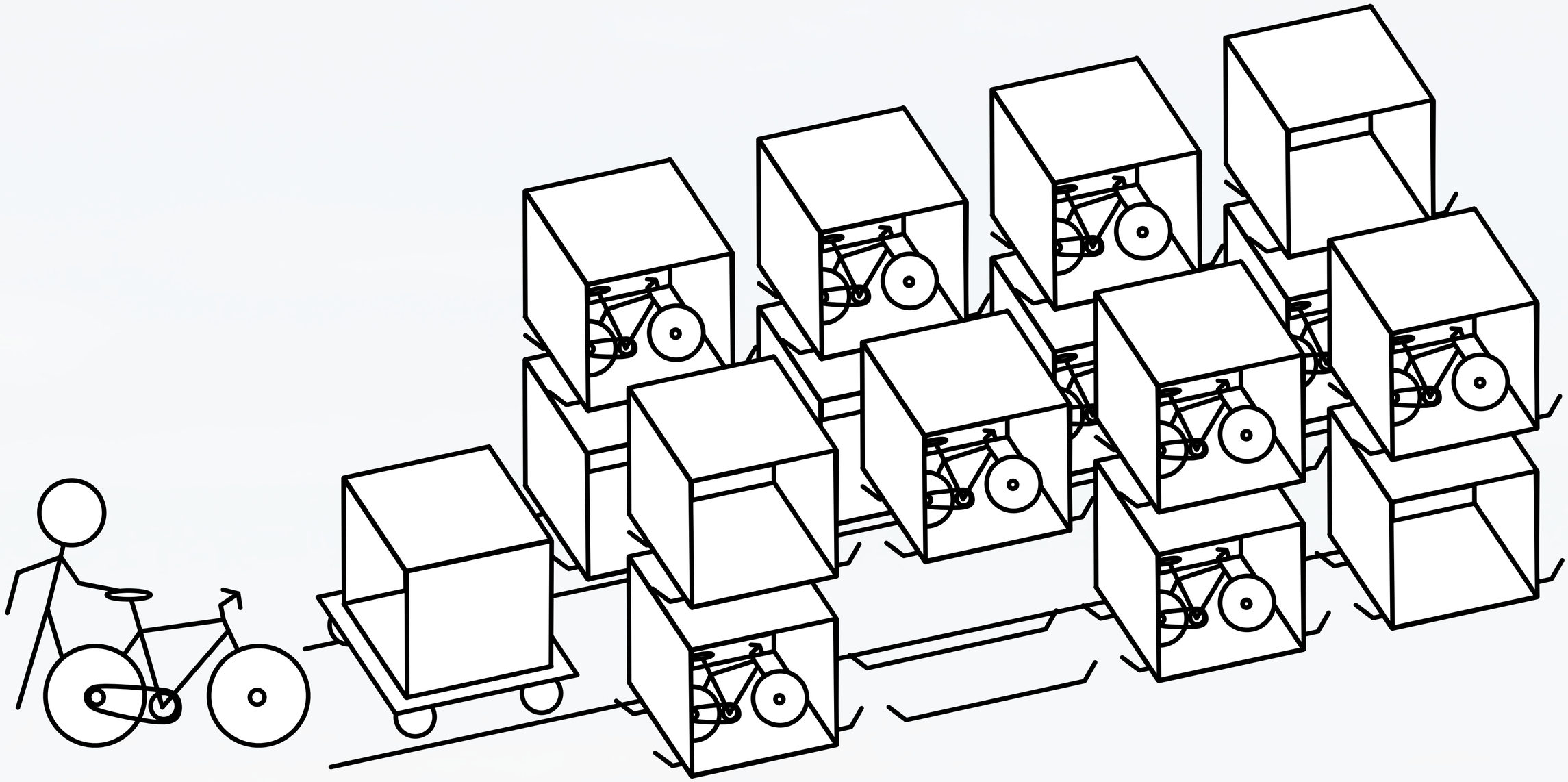
# Bike Loft

1. Introduction
1.1 Motivation



# Overview



Scheduling problem

- How do we efficiently serve the customers?

Sorting problem

- How do we efficiently rearrange the stored boxes?

Survey of other bicycle storage systems

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# Scheduling



# Scheduling



Given:

- robot specification
- customer arrival and departure times

Wanted:

- customer-to-slot-and-door-assignment
  that does not keep the customers waiting

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# Sorting

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# Sorting

1. Introduction

1.2 Overview

# Sorting



Given:

- a graph that describes the layout
- an initial permutation of the boxes
- starting position of the robot

Wanted:

- shortest possible sorting walk

# Contributions



Scheduling Problem

- algorithms for arrival-only with a single door
- *NP*-hardness of the two-door problem
- *NP*-hardness of the all-day problem

Sorting Problem

- algorithms for sorting on paths, trees and cycles
- *NP*-hardness for planar graphs
- *NP*-hardness for trees when minimizing swaps

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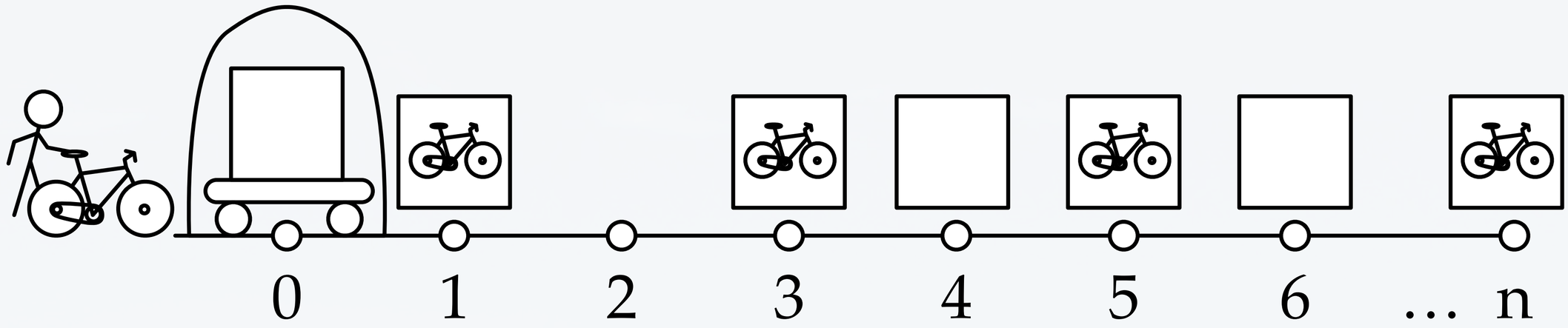
# Scheduling Problem

# Model






Storage System Model

- one-dimensional layout
- $n$ slots that can store a single box each
- initially all boxes are empty
- single door for customer interaction

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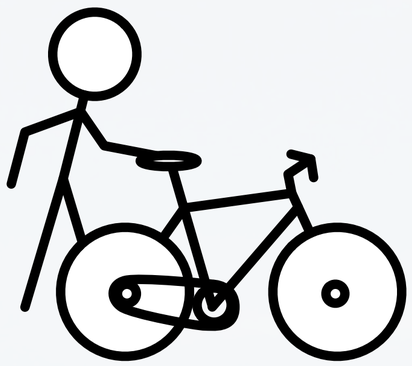
# Customer Model

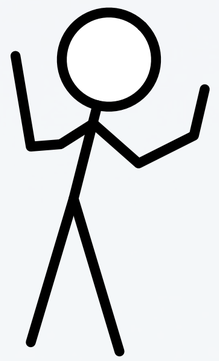


$t_i^a$



arrival

$t_i^d$



departure

Customer Model

- $m$ customers, each with arrival time $t_i^a$ and departure time $t_i^d$ that are known in advance

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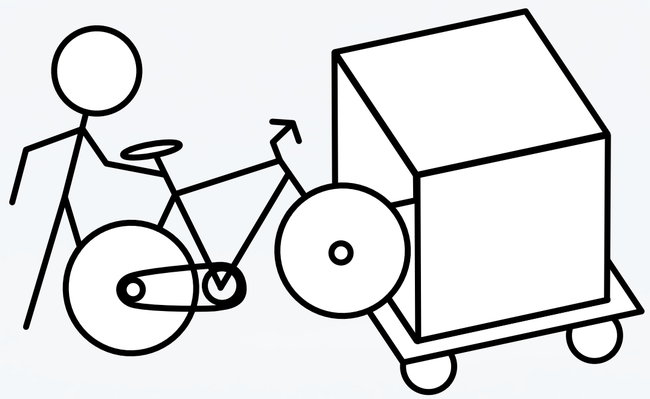
# Customer Model

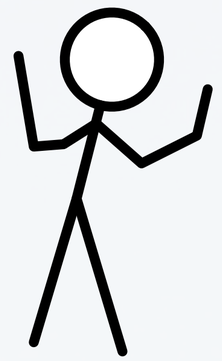


$t_i^a$



arrival

$t_i^d$



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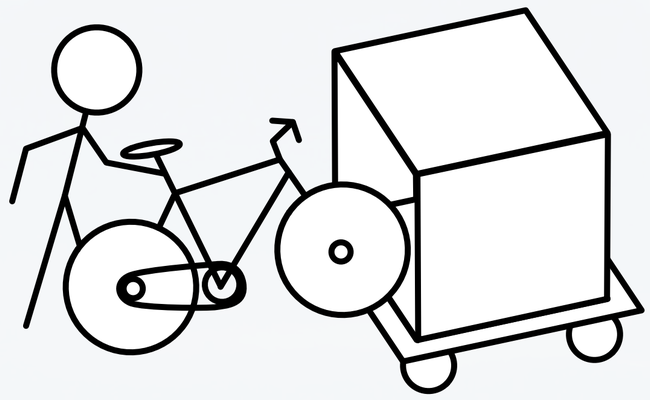
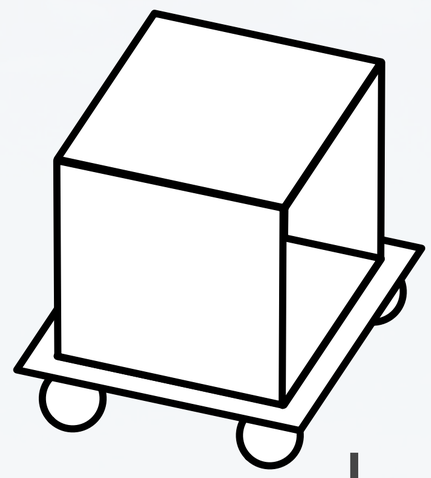
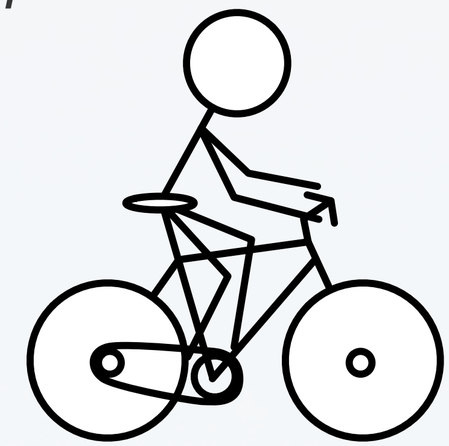


arrival



$t_i^d$



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Customer Model

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# Robot Model



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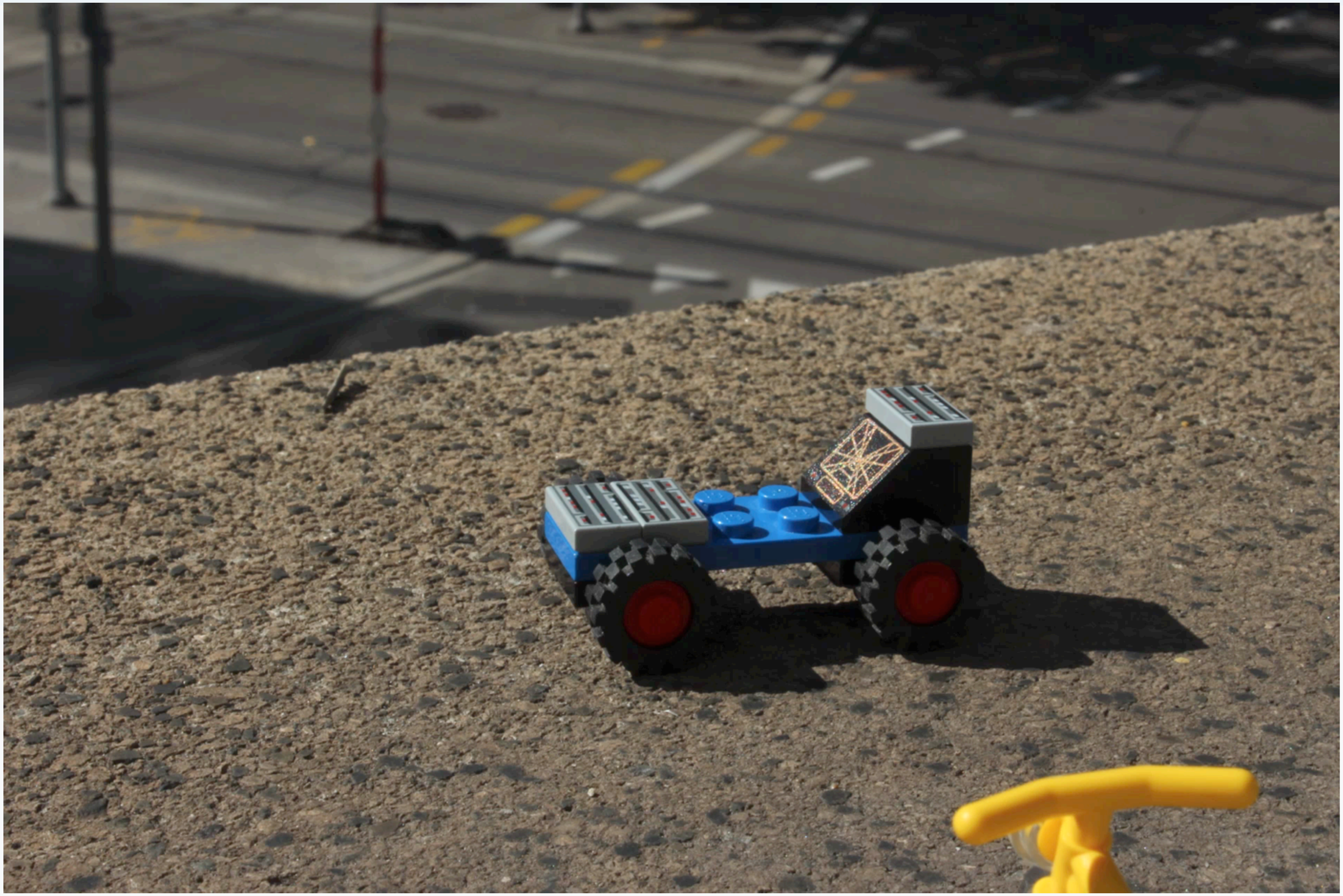
# Robot Model

2. Scheduling

2.1 Model



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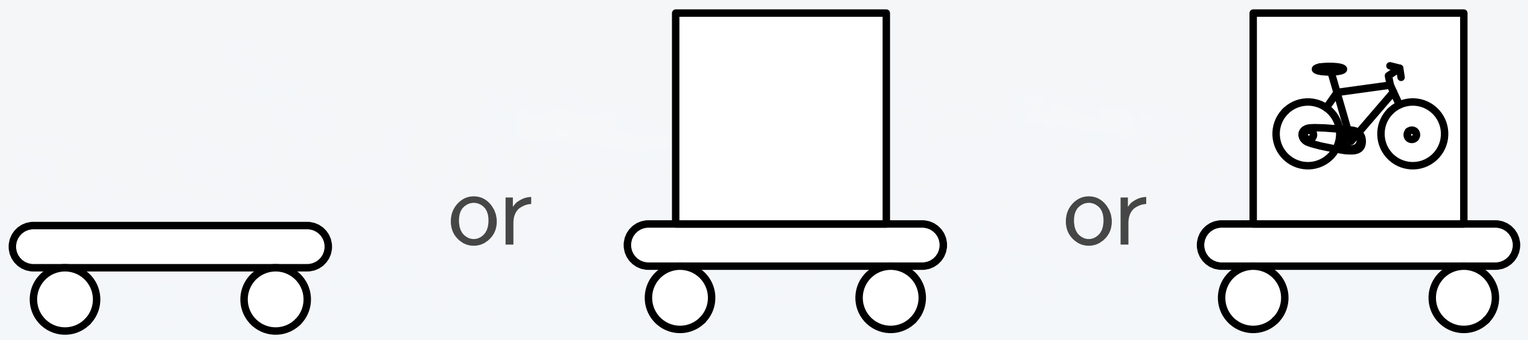
# Robot Model



# Robot Model





## Robot Model

- a single box at a time
- unit velocity, no acceleration
- instant box handling
- starts at the door
- distinction: can boxes be swapped or not?

# Model



Goal

- no customer should ever wait at the door
  (neither for an empty box nor for his bike)

Task

- find a robot schedule that achieves this

Assumption

- boxes not rearranged while in storage

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# Arrival Only



Early morning scenario - no departures yet.

# Arrival Only



Every slot can be used for at most one customer.

If $n = m$, we can describe the schedule by a permutation $\pi \in S_n$.

The box of customer $i$ gets stored at slot $\pi(i)$.

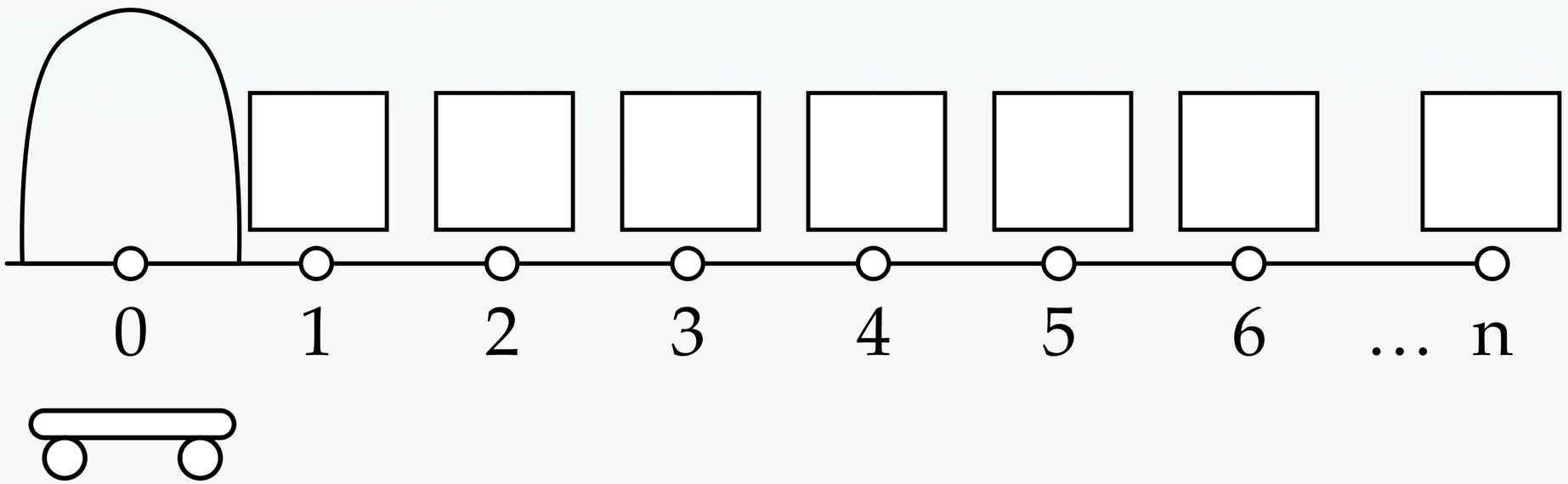
Which permutations correspond to feasible, wait-free robot schedules?

# No Swapping



No swaps and initially no box on the robot.

How to serve the first customer?



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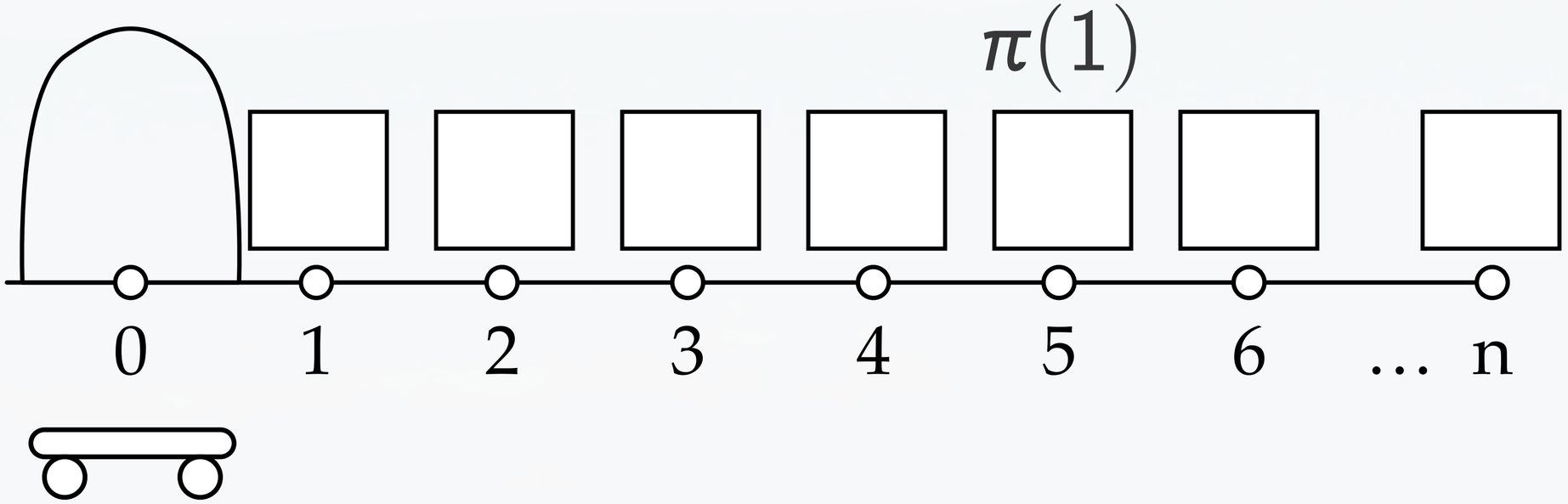

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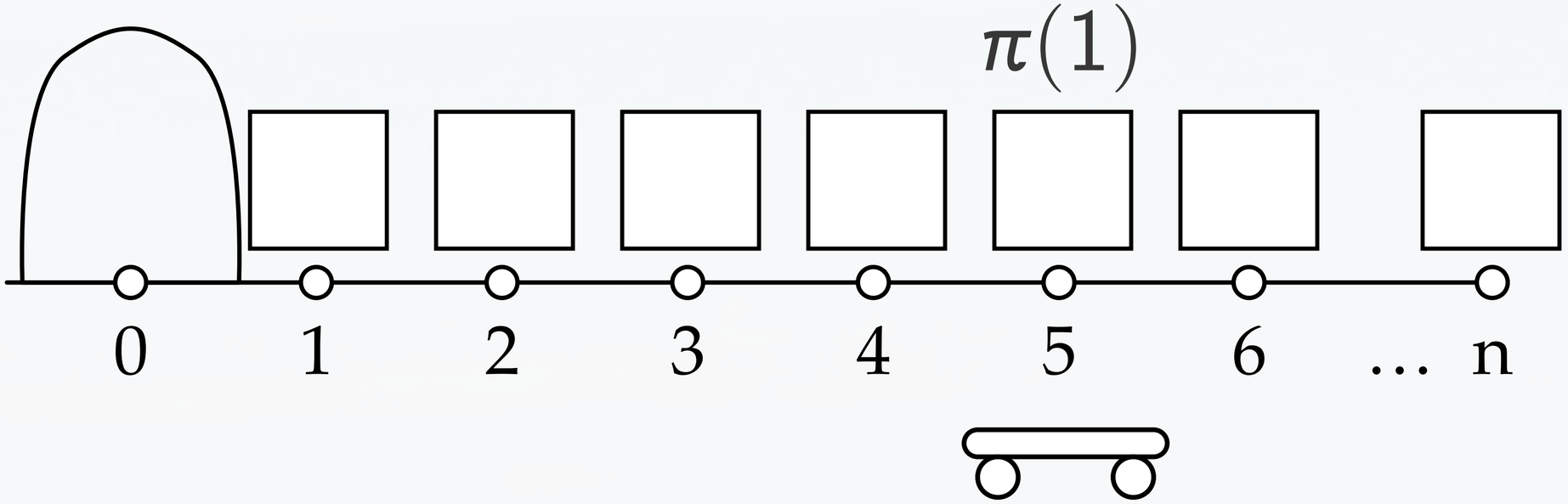

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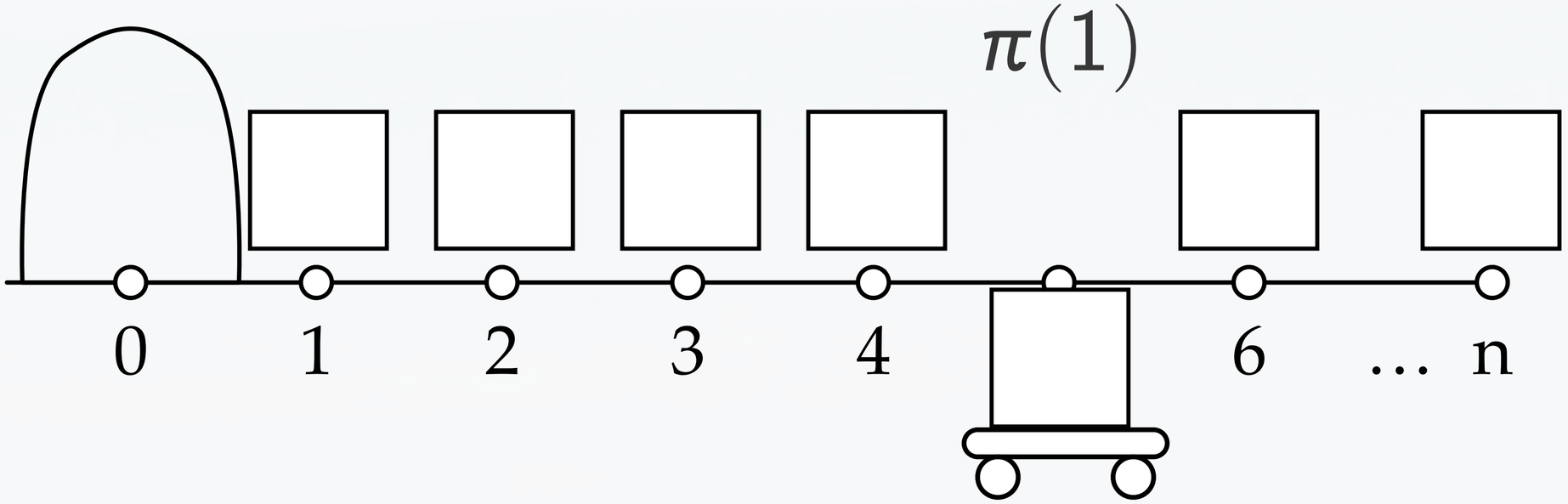

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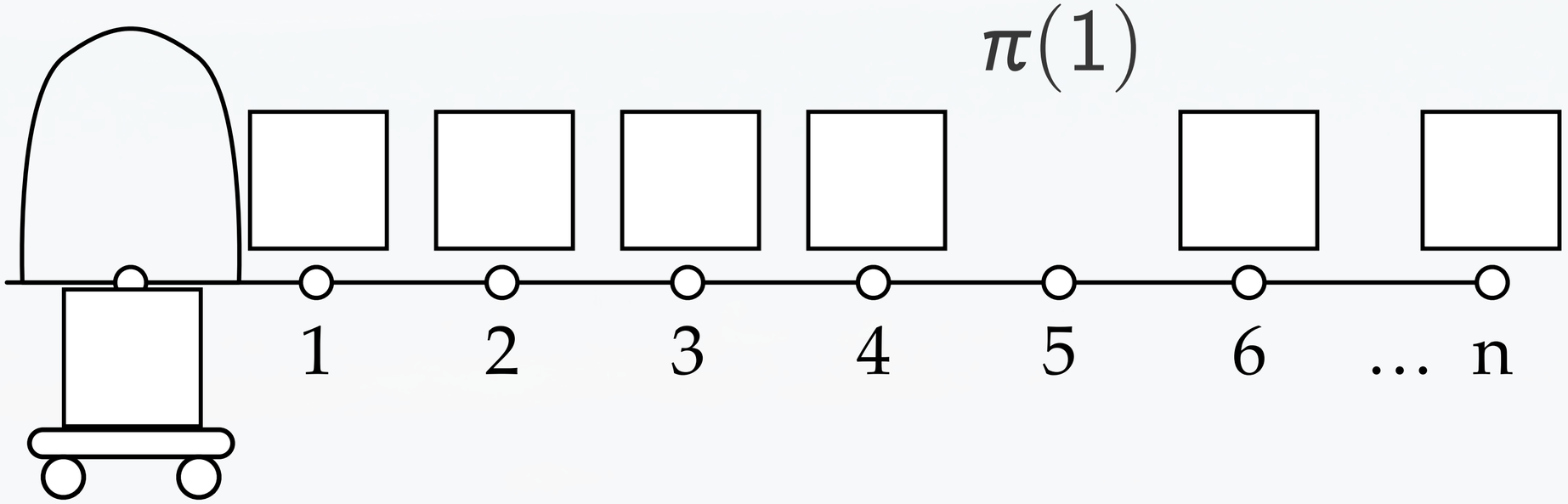

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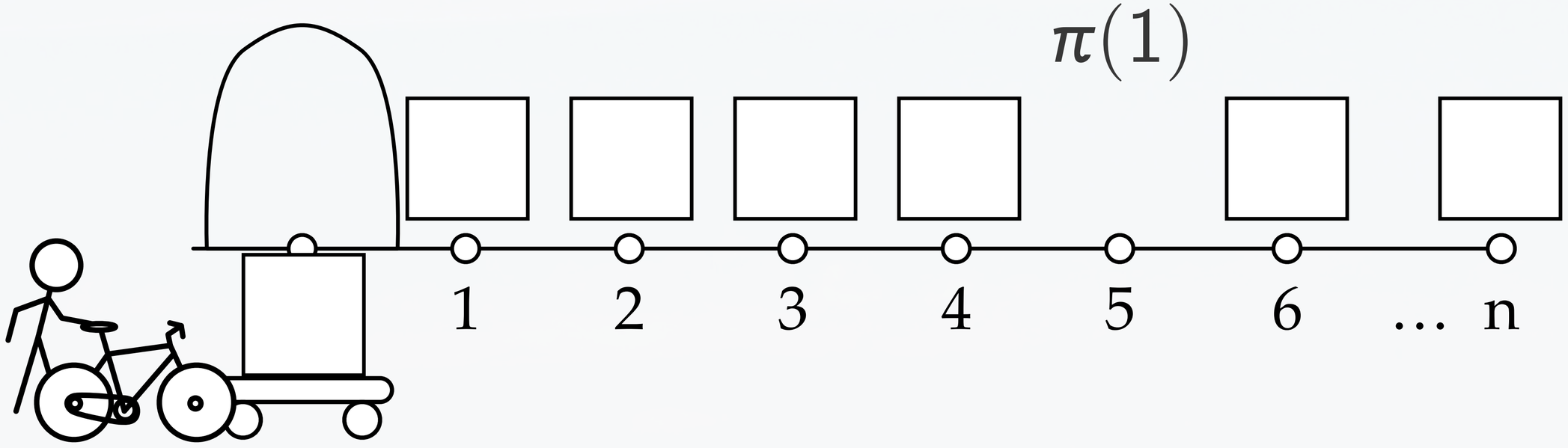

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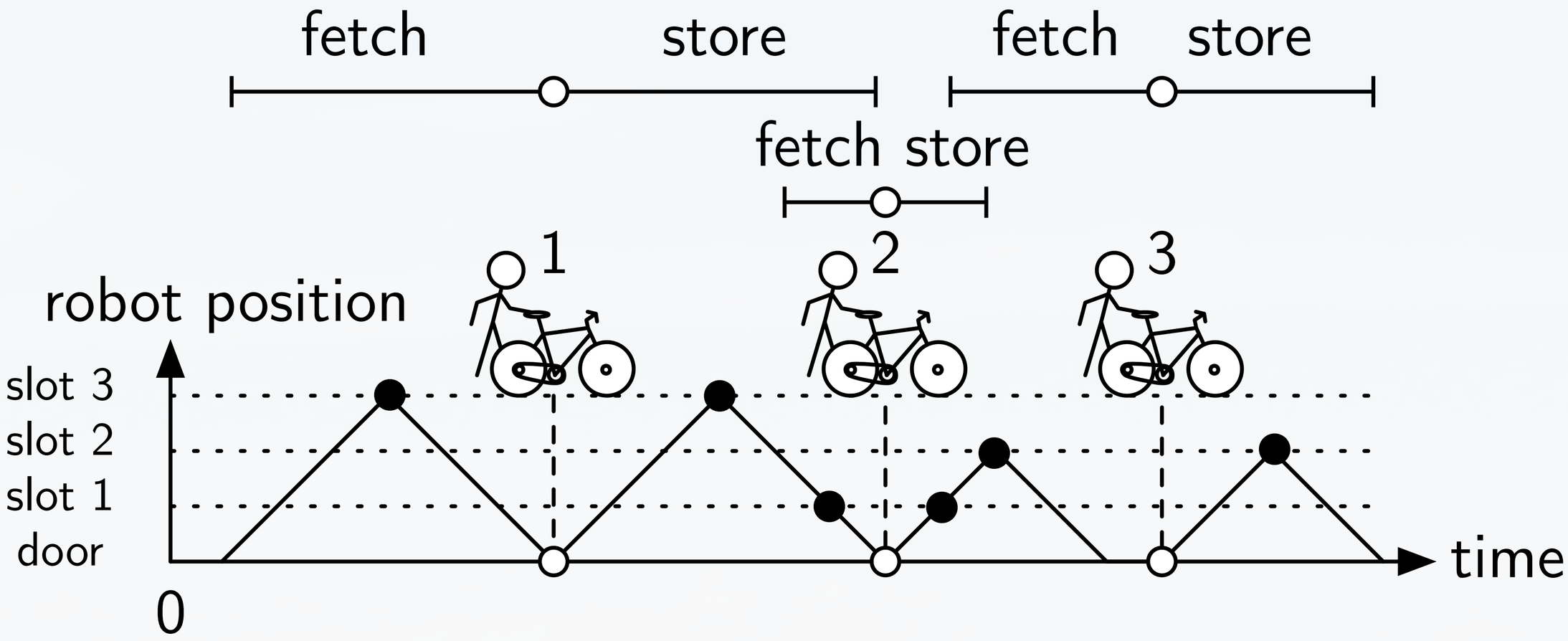
Example:



# No Swapping



Between the arrivals of customers *i and i+1*:

- store the full box at slot $\pi(i)$
- fetch the empty box from slot $\pi(i+1)$.

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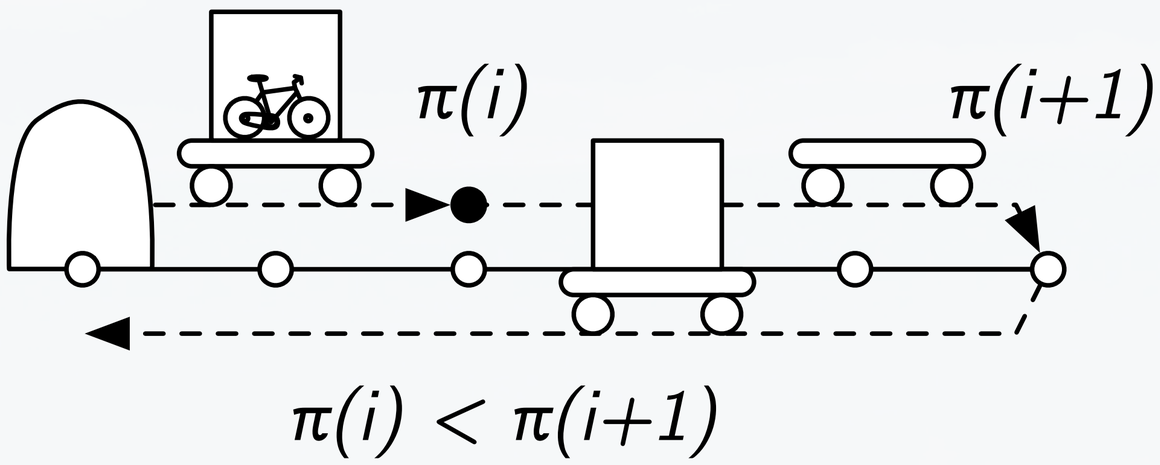


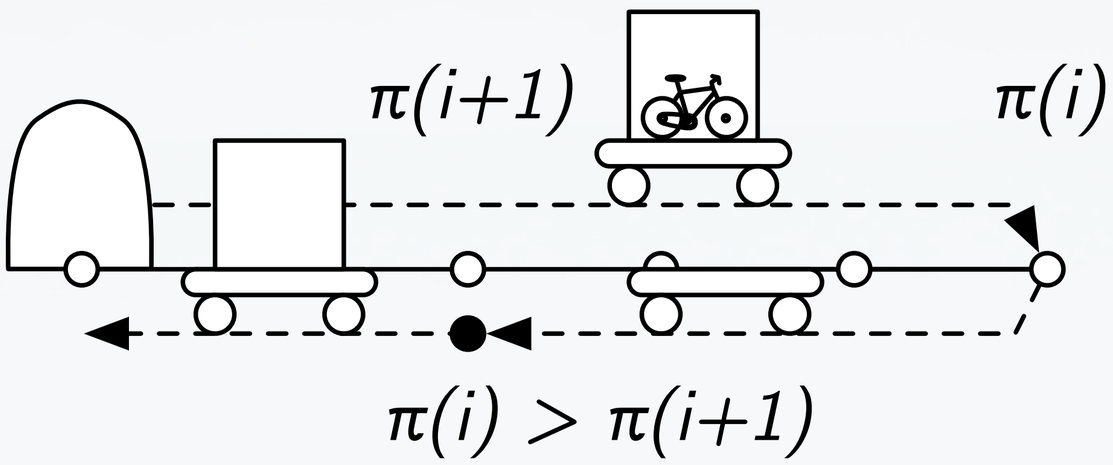


In either case, the robot needs $2 \cdot \max(\pi(i), \pi(i+1))$.

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# No Swapping



*Definition:* ***Inter request times***

Let $\Delta = \{\Delta_0, \Delta_1, \ldots, \Delta_n\}$, where $\Delta_i = t^a_{i+1} - t^a_i$.

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*Definition:* **No-Swap Arrival-Only *problem***

Given $\Delta$, find $\pi$ with $2 \cdot \max(\pi(i), \pi(i+1)) \leq \Delta_i$.

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Given $\Delta$, find $\pi$ with $2 \cdot \max(\pi(i), \pi(i+1)) \leq \Delta_i$.

*Equivalently:*

Given $\Delta$, find $\pi$ with $\pi(i) \leq \min\left(\frac{\Delta_{i-1}}{2}, \frac{\Delta_i}{2}\right)$.

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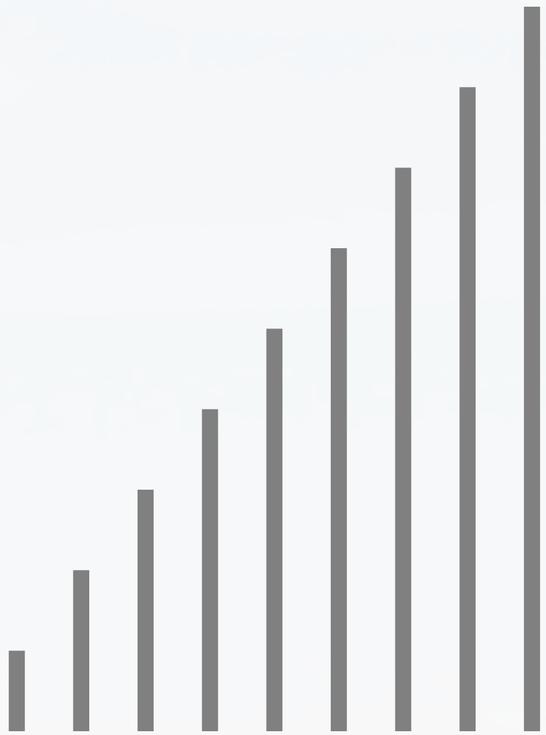
# Constrained π

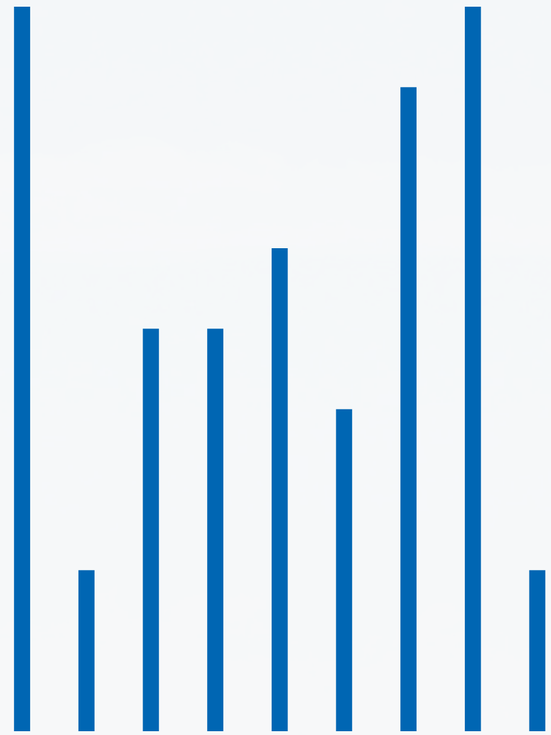


*Definition:* **Constrained Permutation *problem***

Given $X = \{x_1, \ldots, x_n\}$, find $\pi \in S_n$ with $\pi(i) \leq x_i$.



$\pi$



$X$

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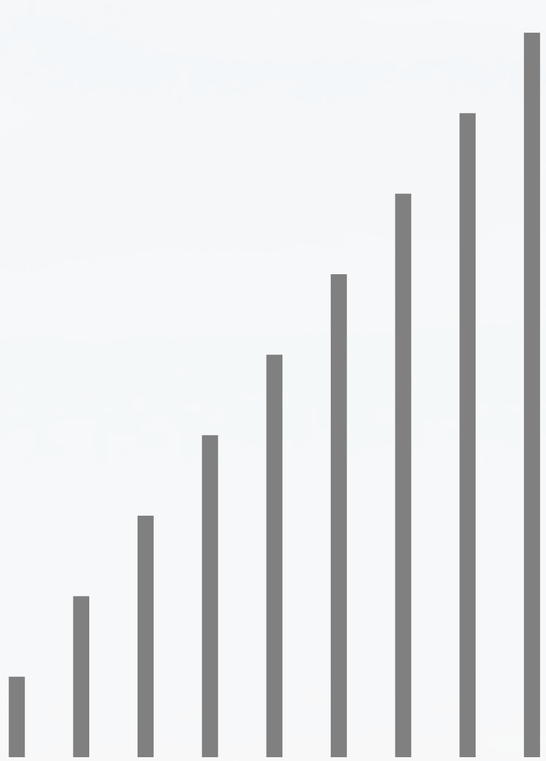

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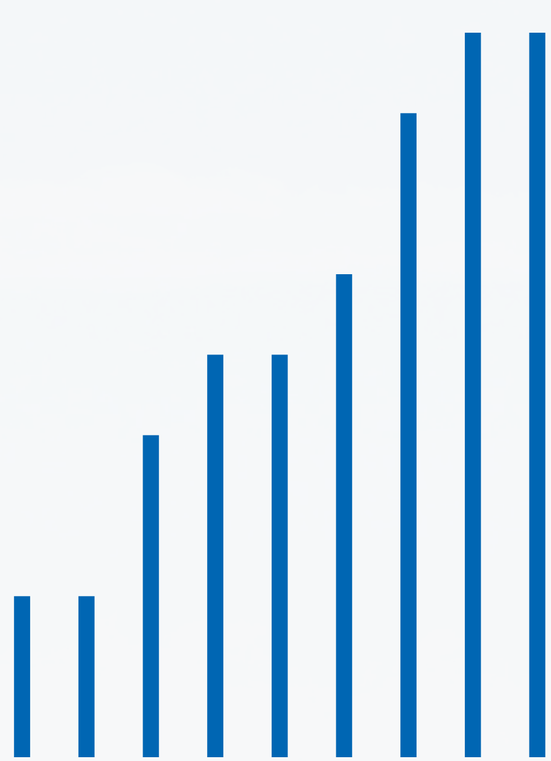

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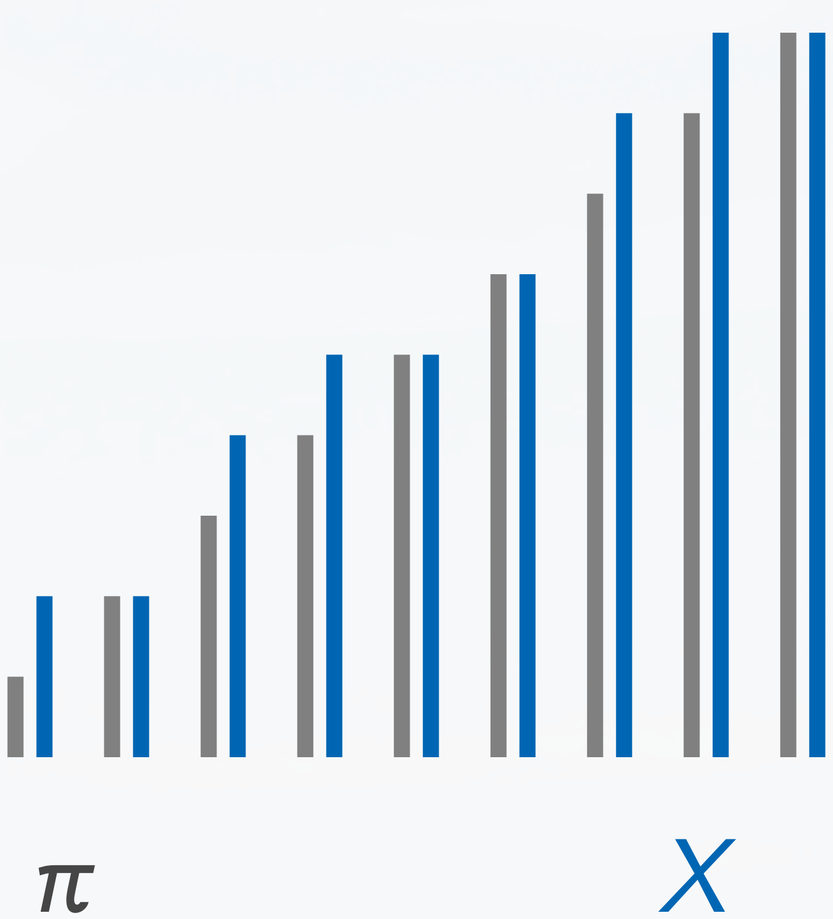

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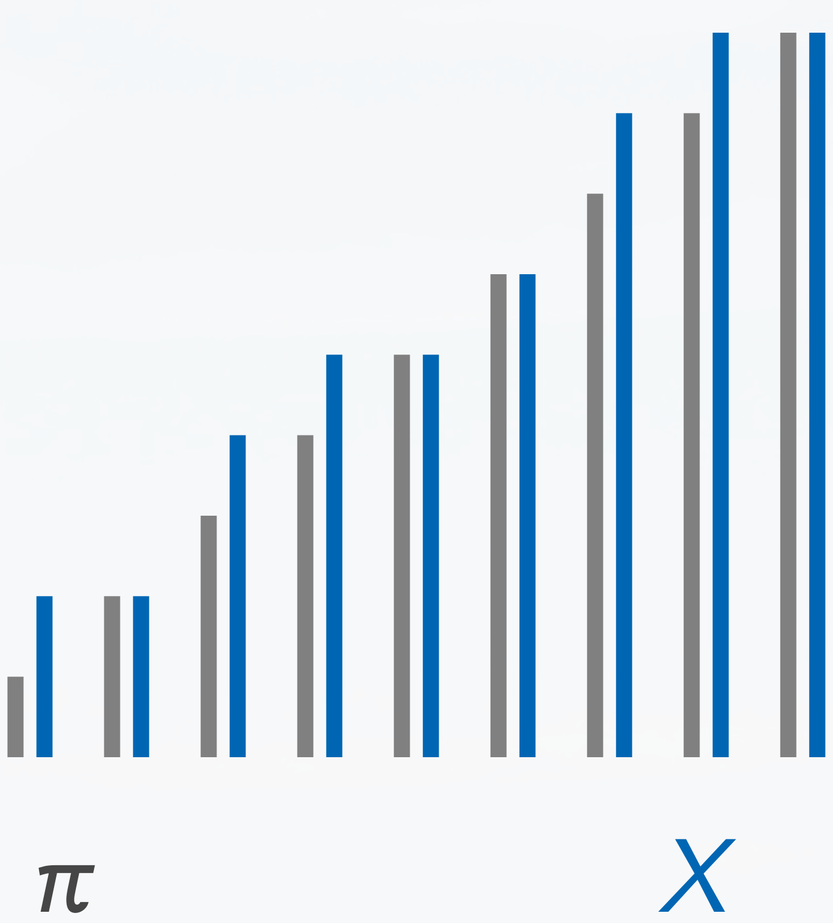


*Solution:*

Sort-and-compare in linear time using bucket sort.

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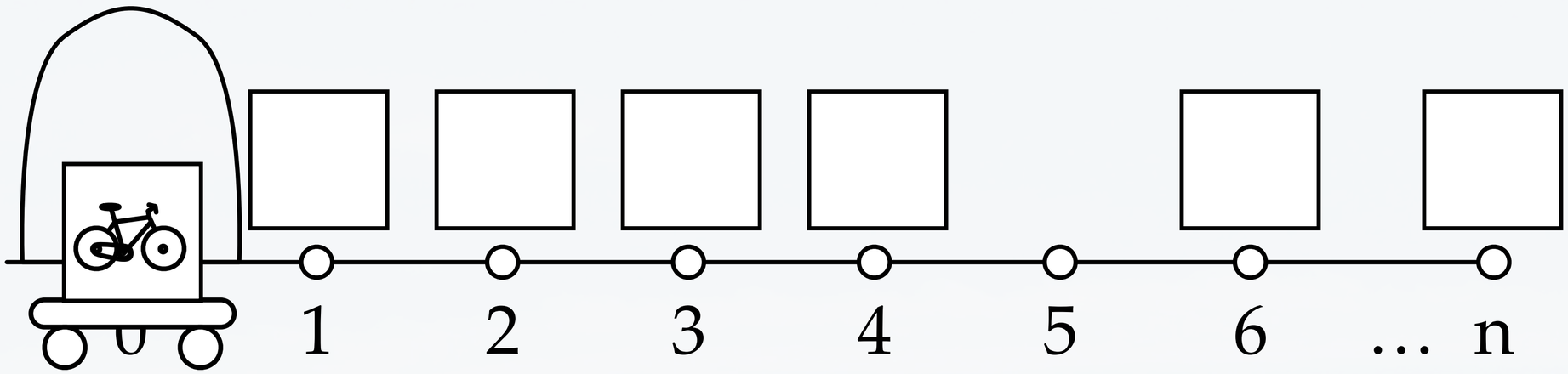
# Return to Door



*Side note:* What if we let the robot always return to the door in order to decouple the requests?



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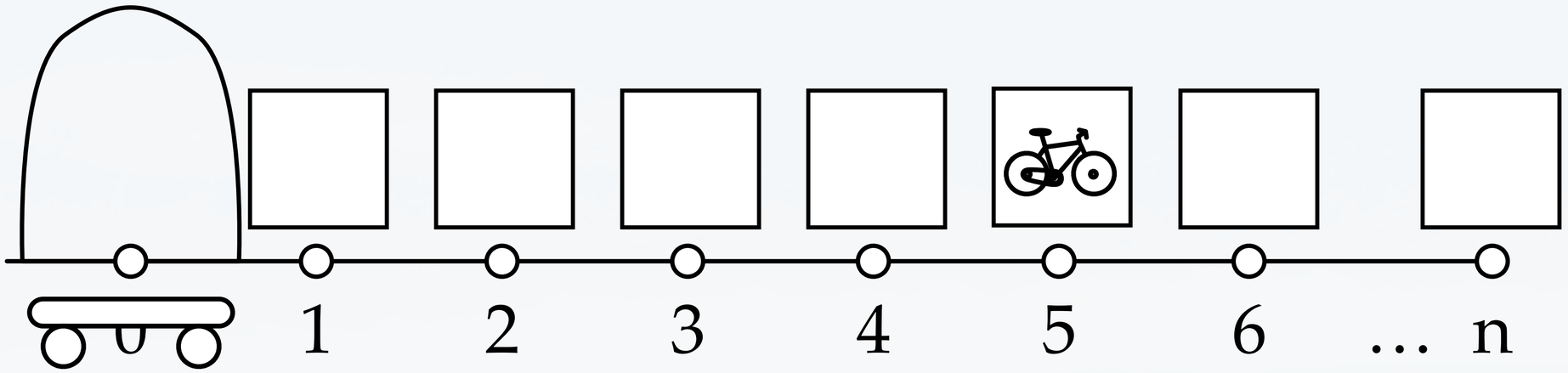

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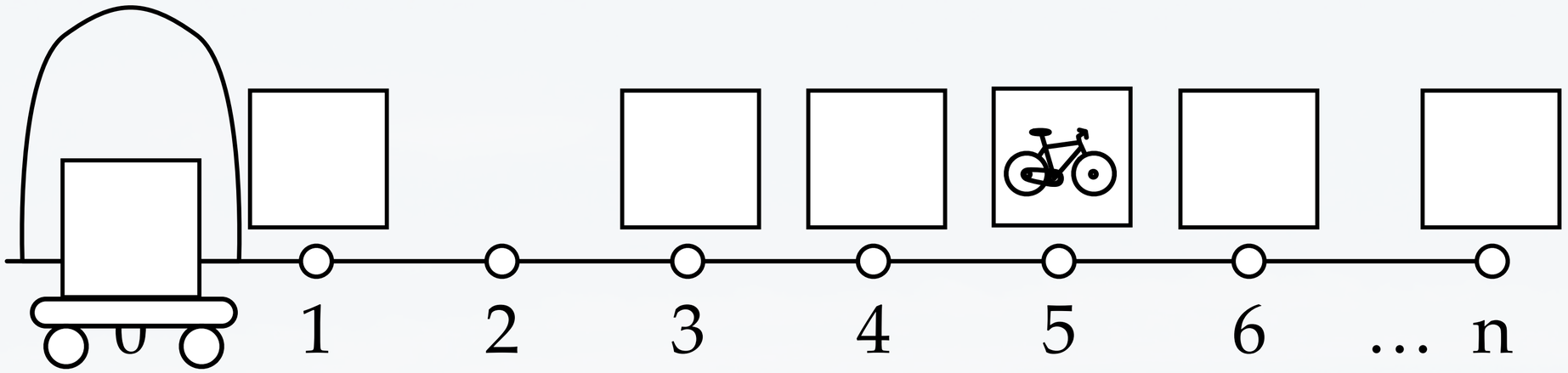

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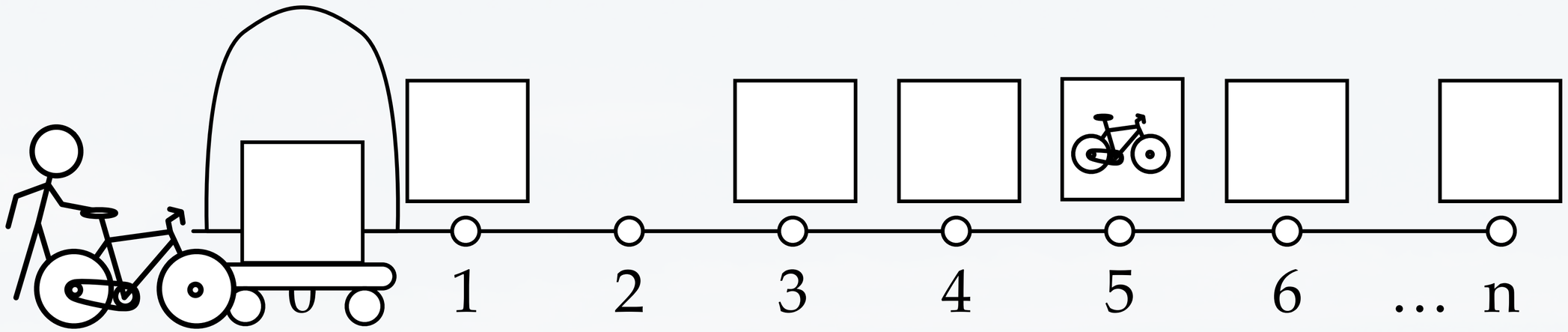

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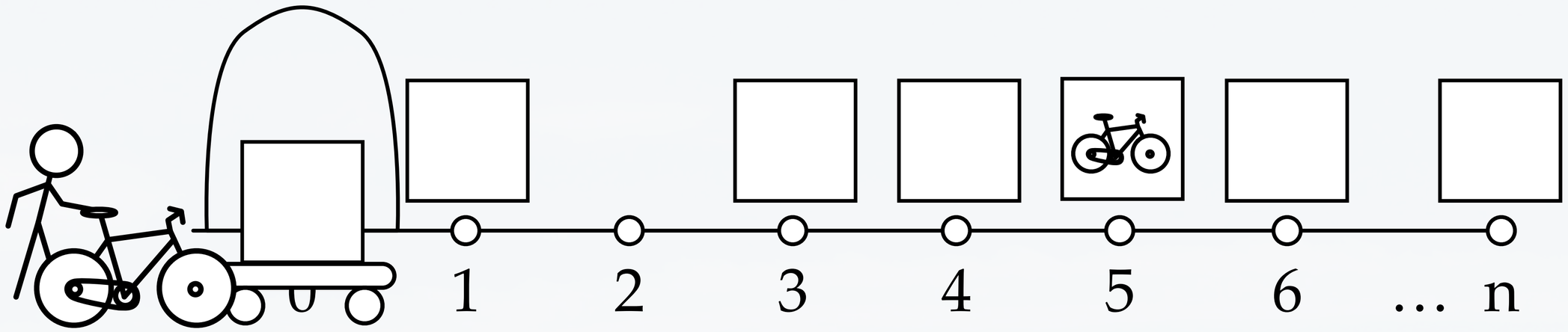


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Given $\Delta$, find $\pi$ with $2\pi(i) + 2\pi(i+1) \leq \Delta_i$.

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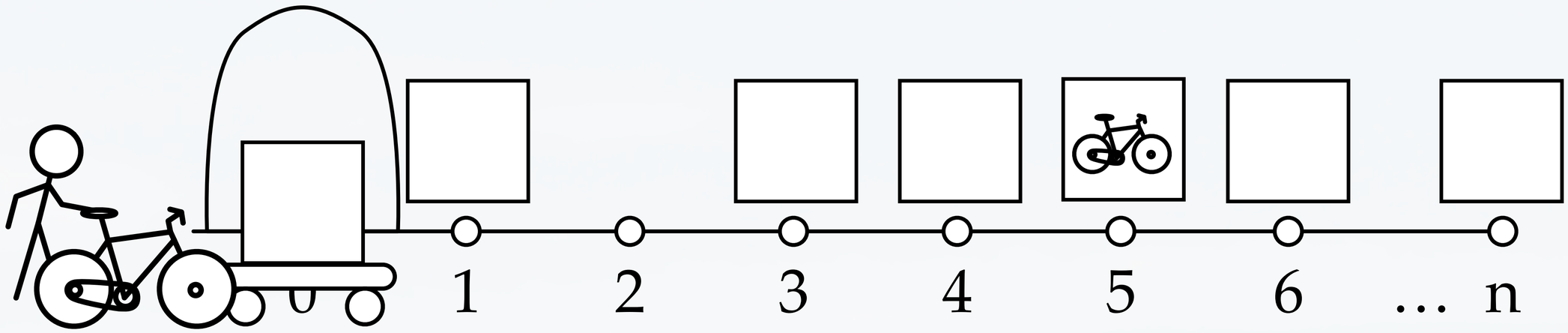


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[2015, Couëtoux, Labourel] Strongly *NP*-complete.

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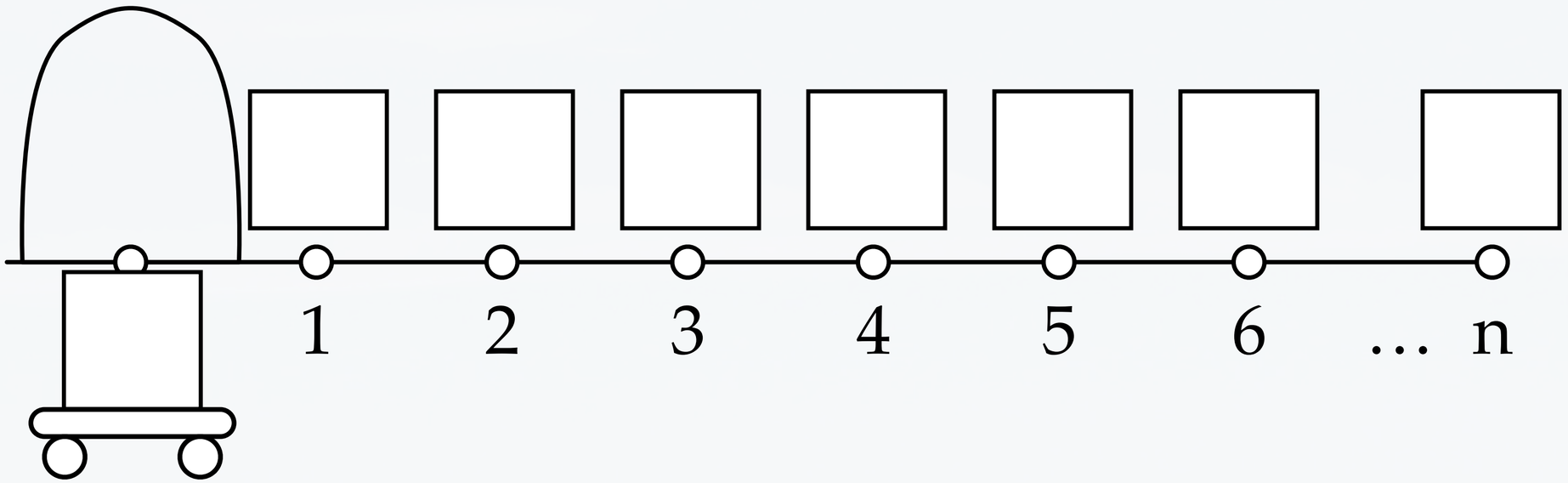
# Box-Swapping



Allow swaps and start with a box on the robot.



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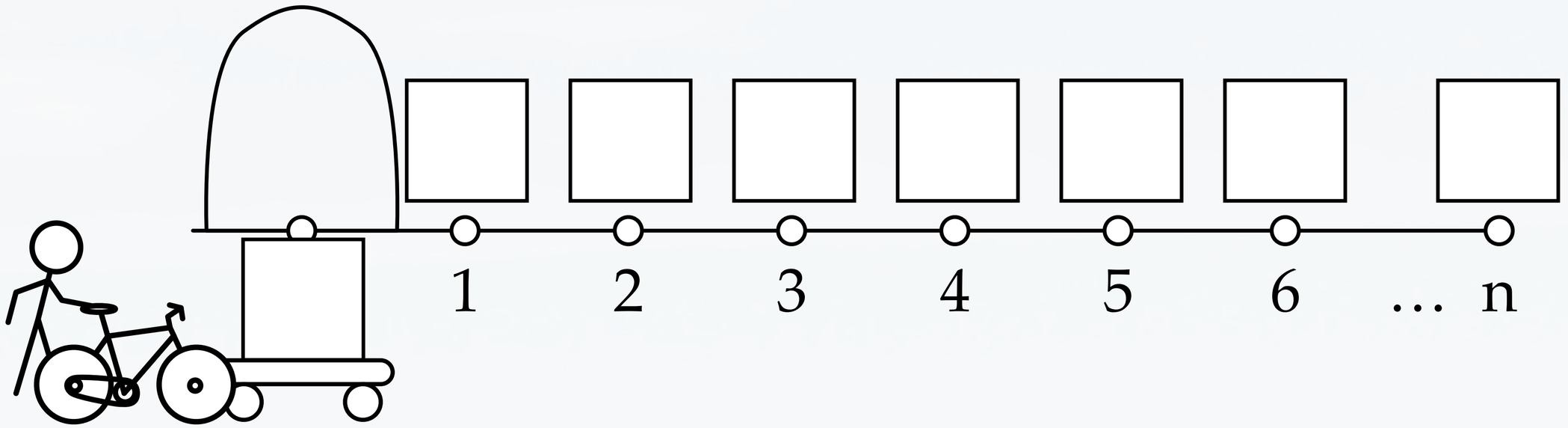

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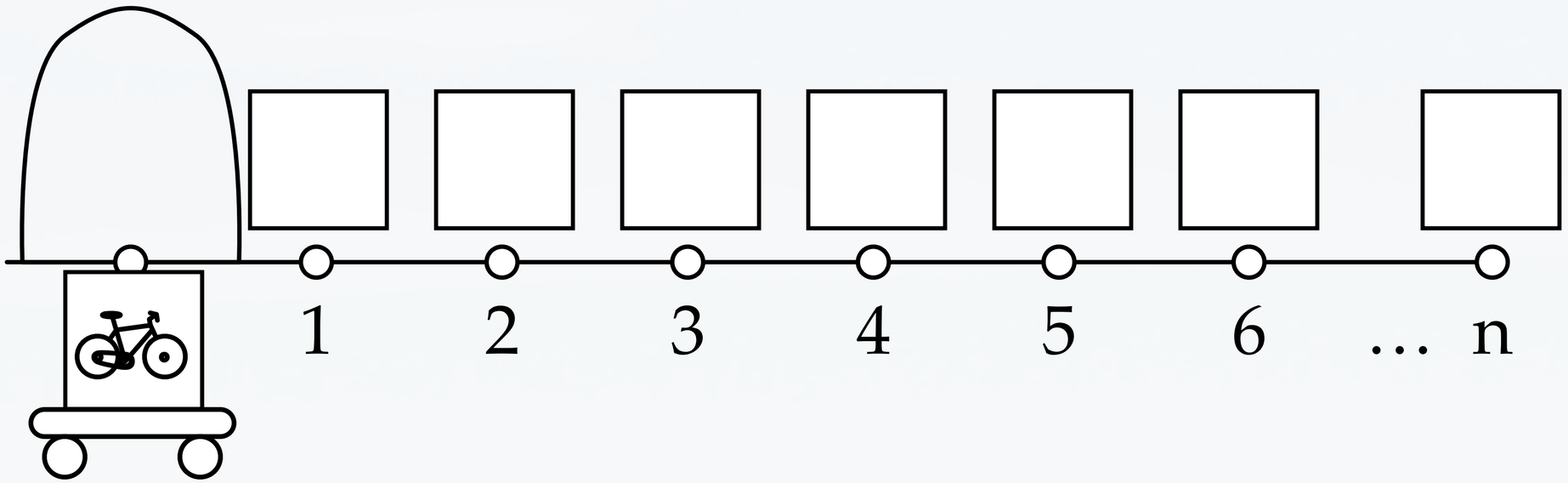

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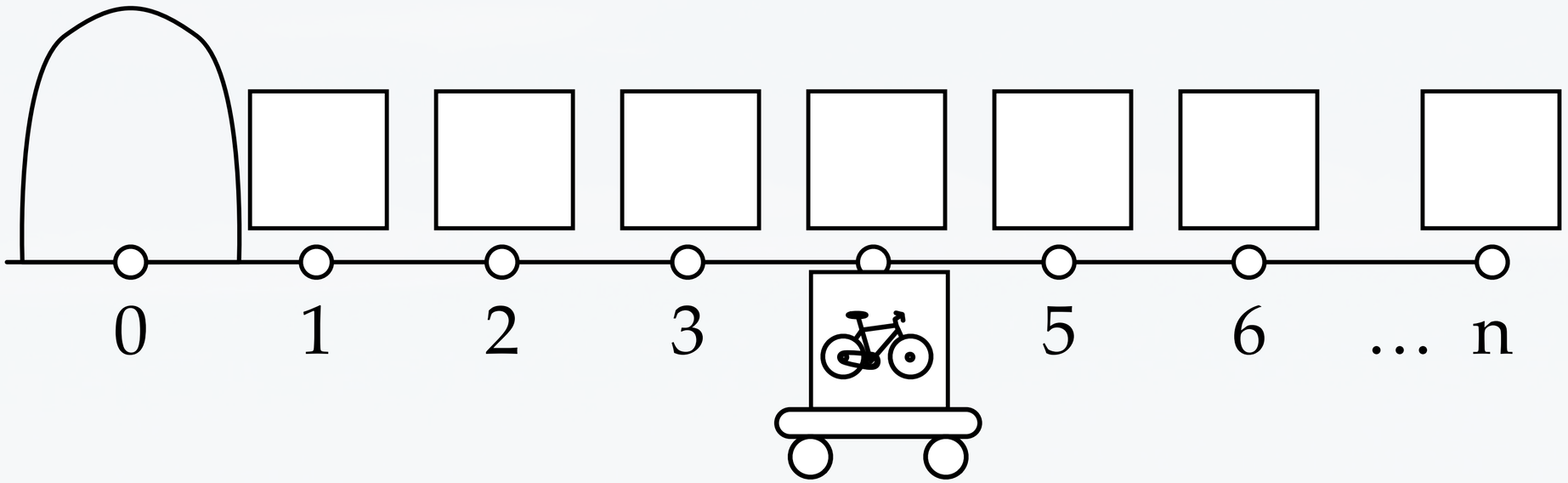

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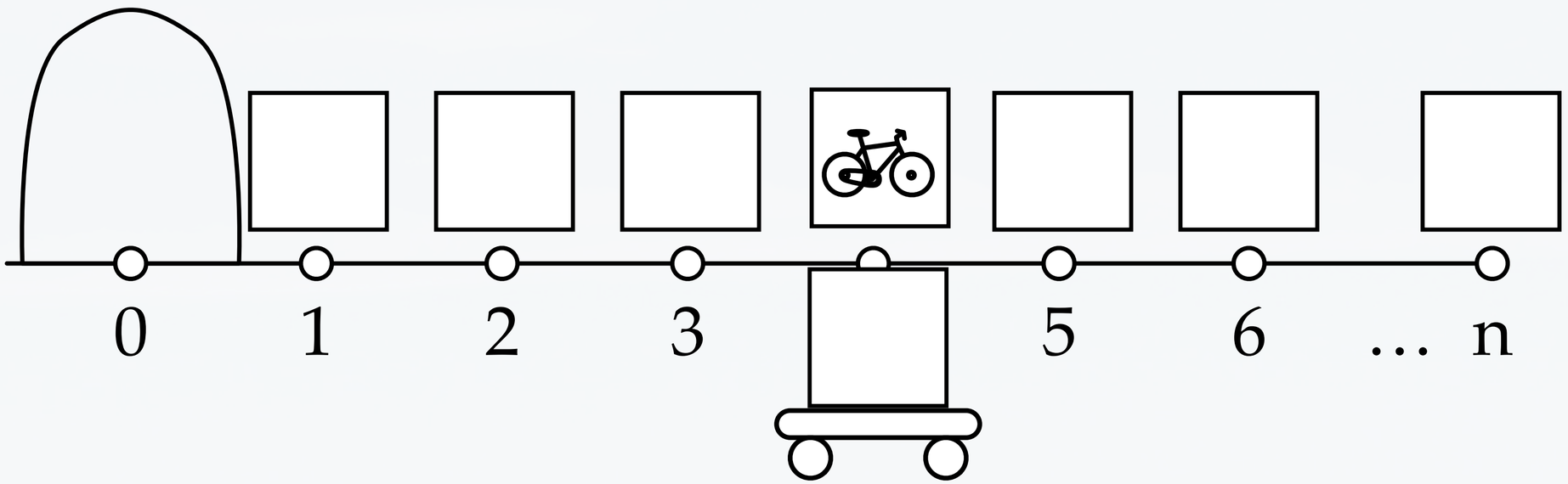
Allow swaps and start with a box on the robot.



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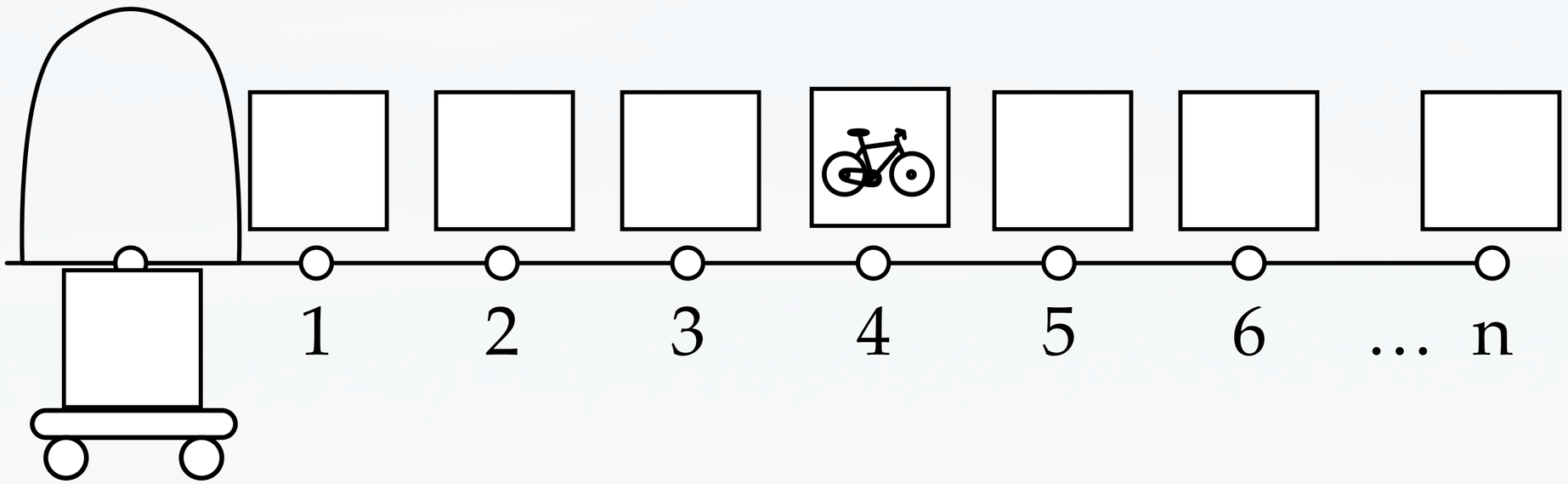

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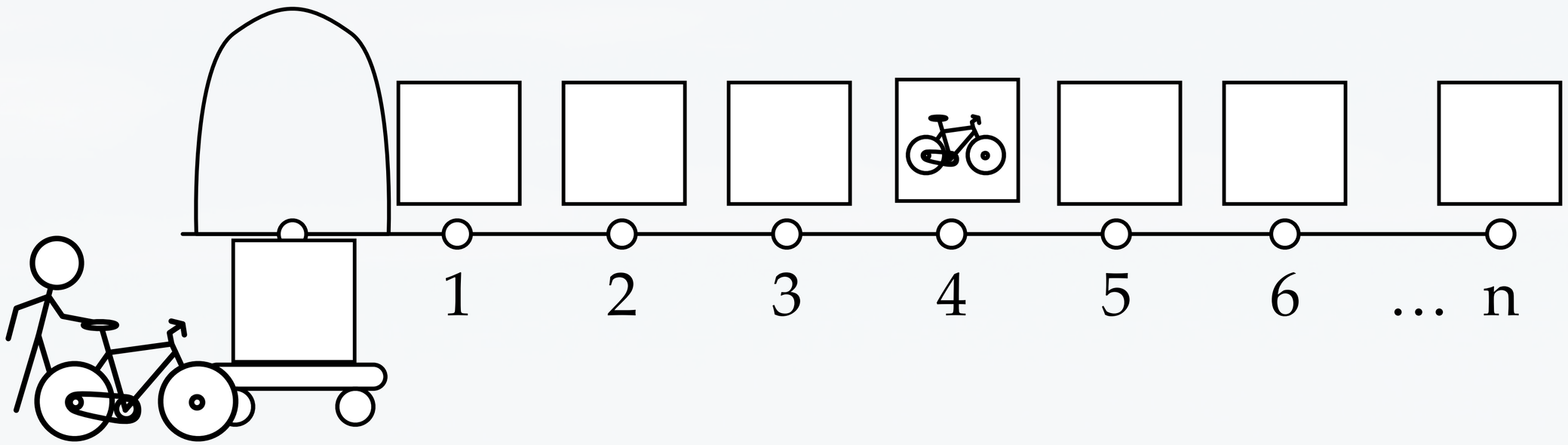

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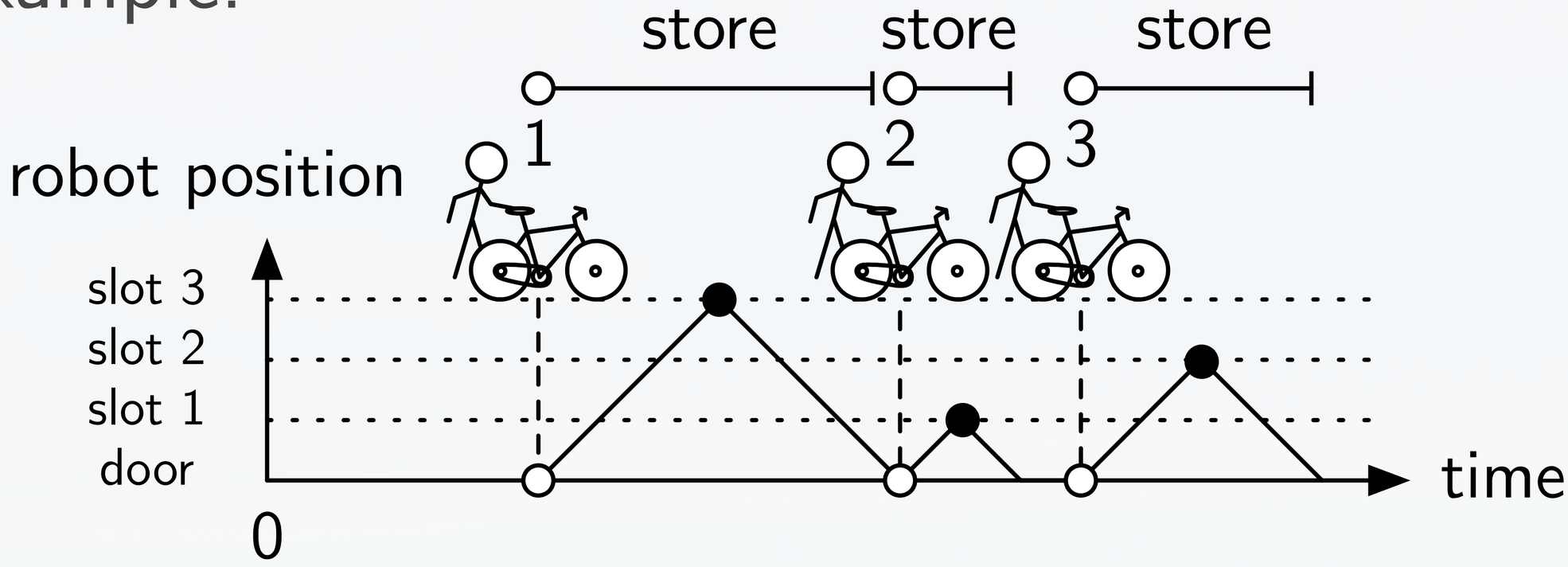
# Box-Swapping



Example:



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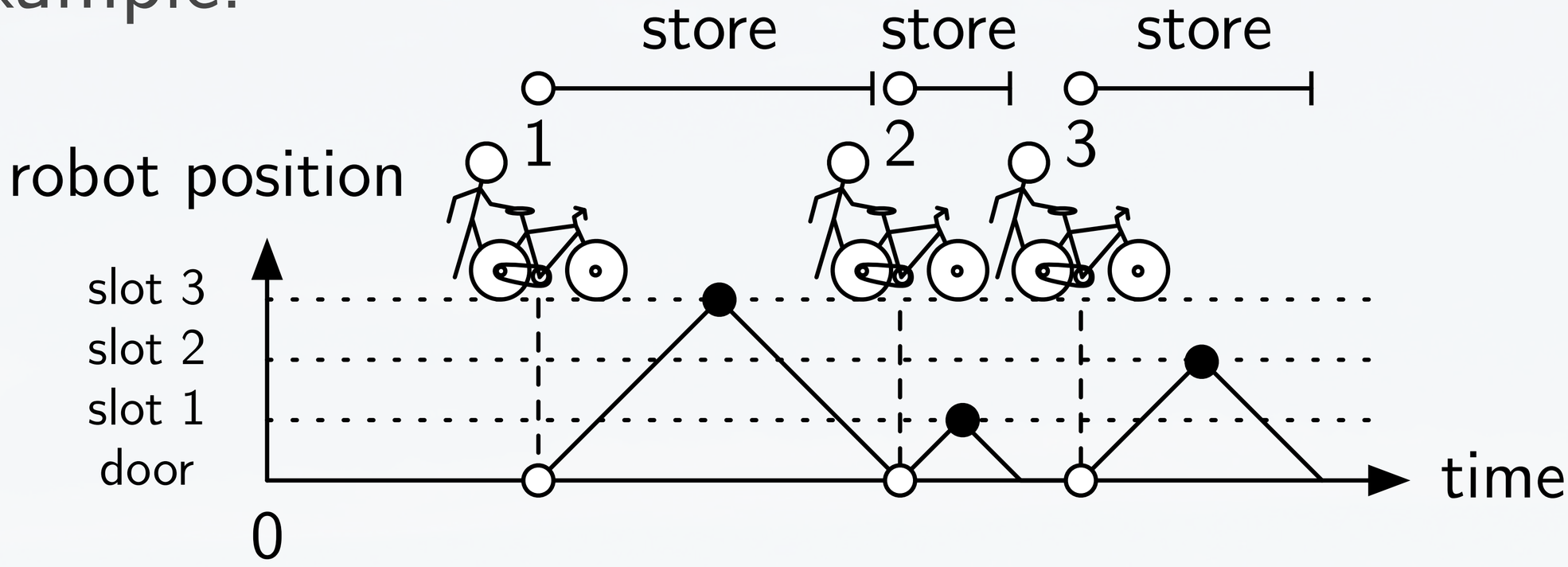
# Box-Swapping



Example:




*Definition:* ***Swap Arrival-Only problem***

Given $\Delta$, find $\pi$ with $2\pi(i) \leq \Delta_i$.

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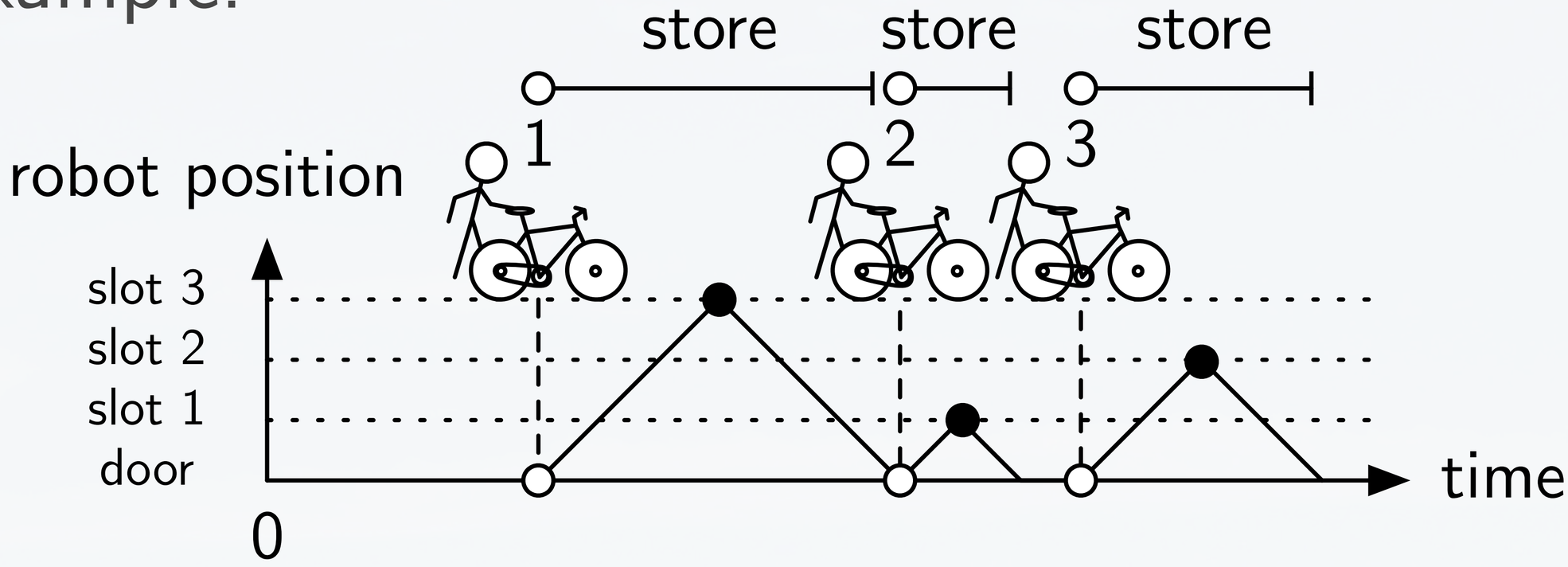
# Box-Swapping



Example:




*Definition:* ***Swap Arrival-Only problem***

Given $\Delta$, find $\pi$ with $2\pi(i) \leq \Delta_i$.

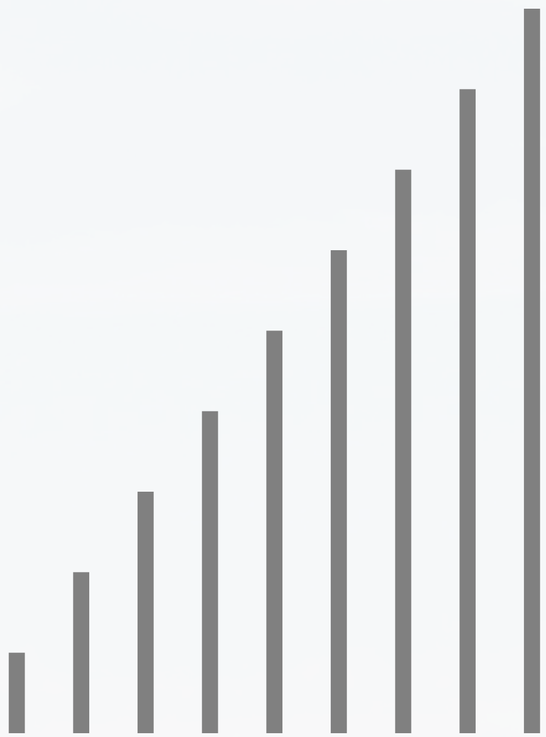
$\rightarrow$ *decide with sort-and-compare*

# Constrained π

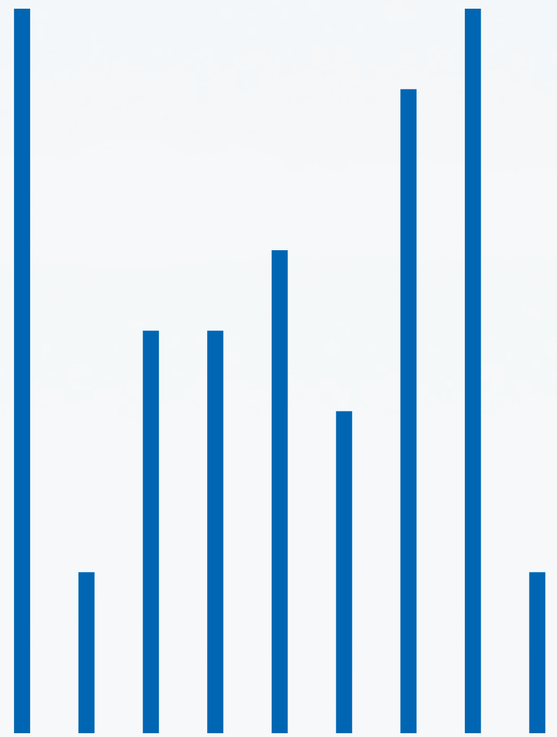


*Alternative approach:* Greedy assignment

Go through the constraints in order and use the largest possible, unassigned permutation entry.



$\pi$



$X$

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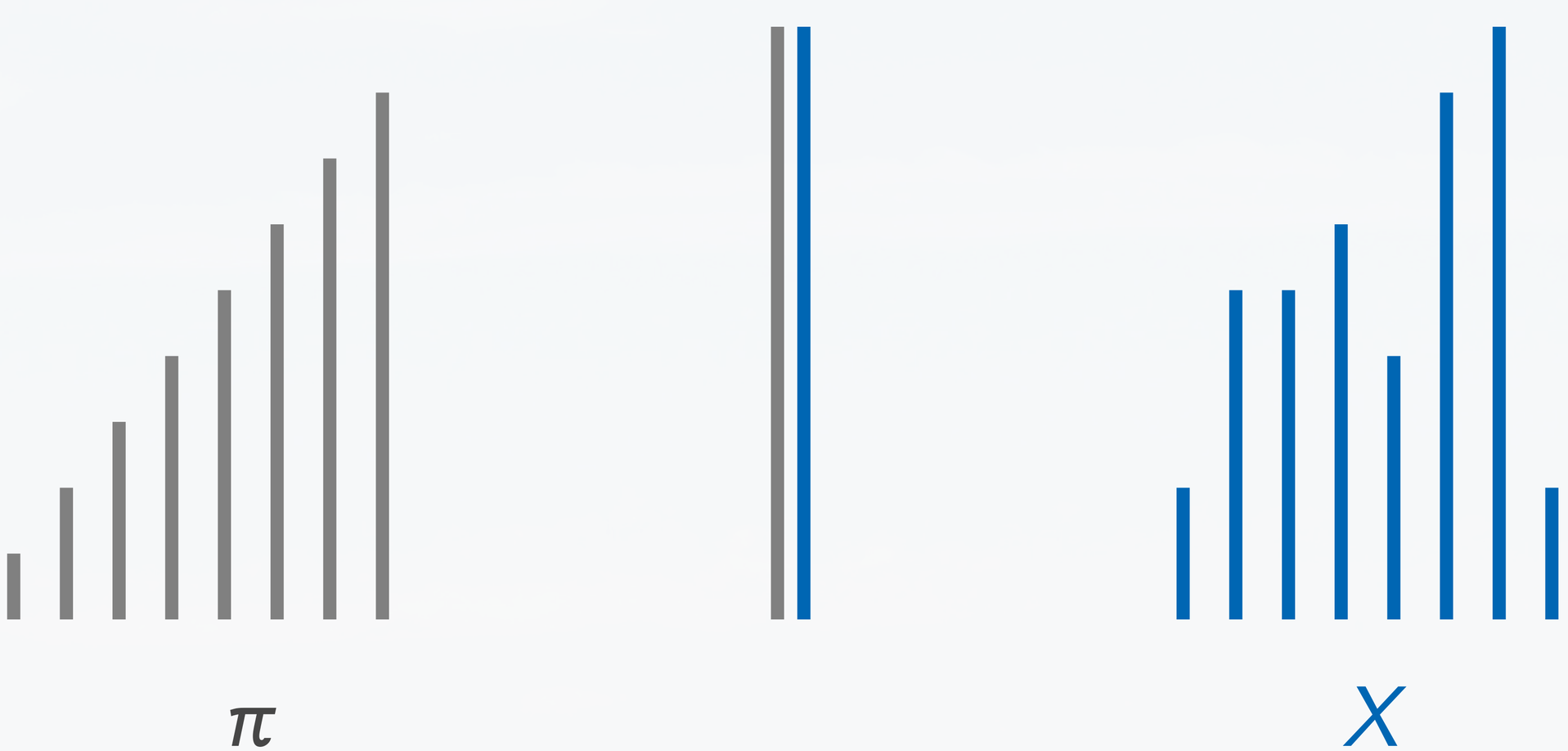

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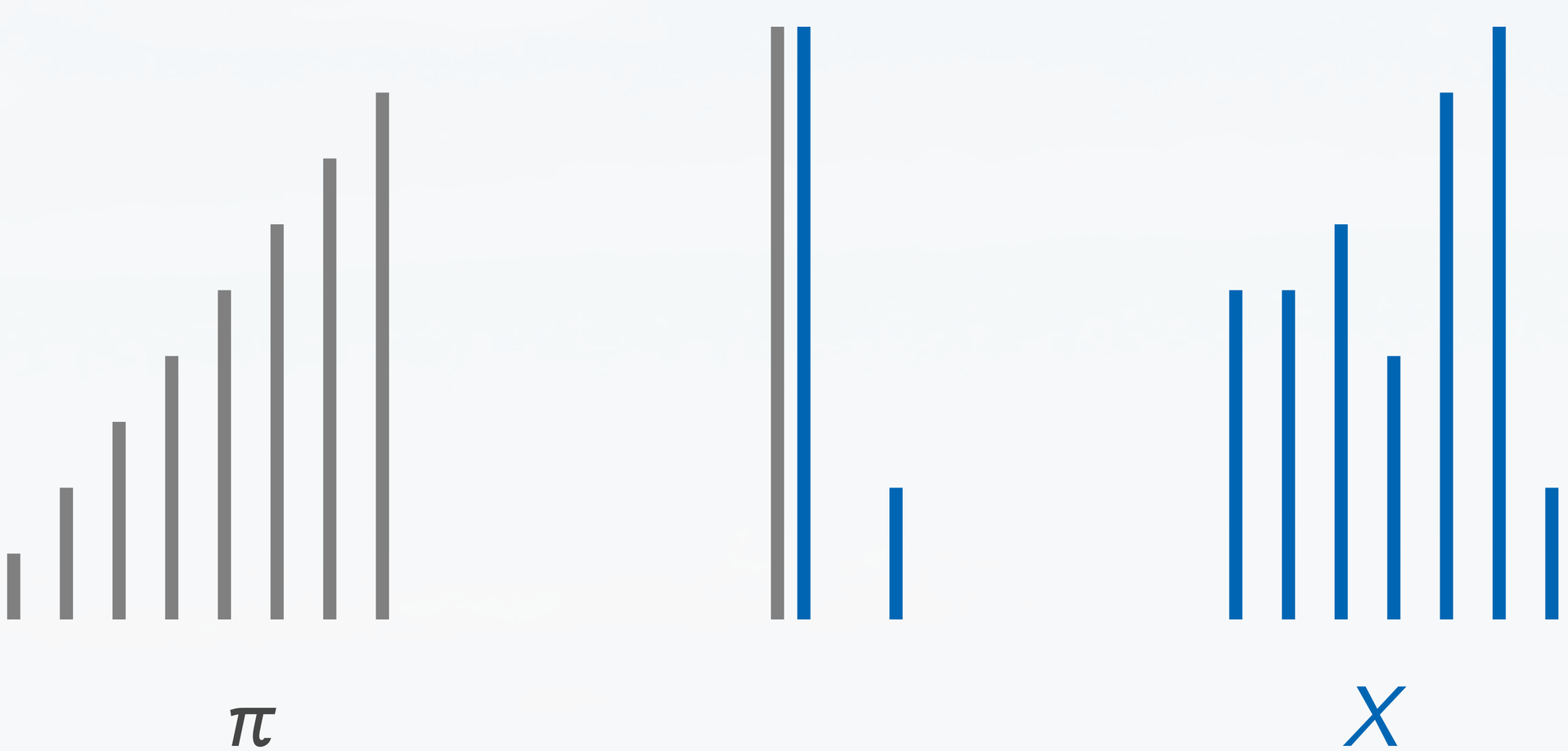

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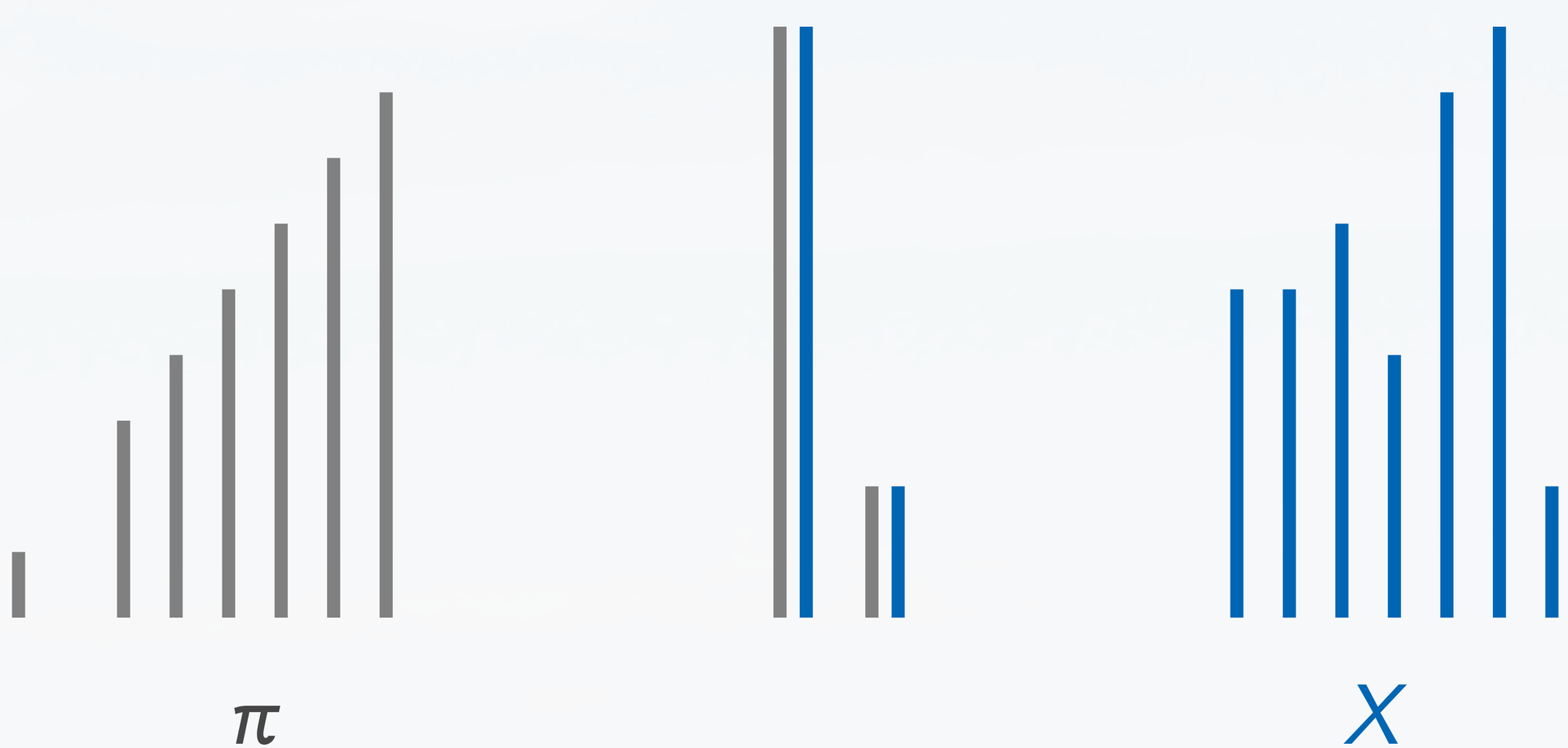

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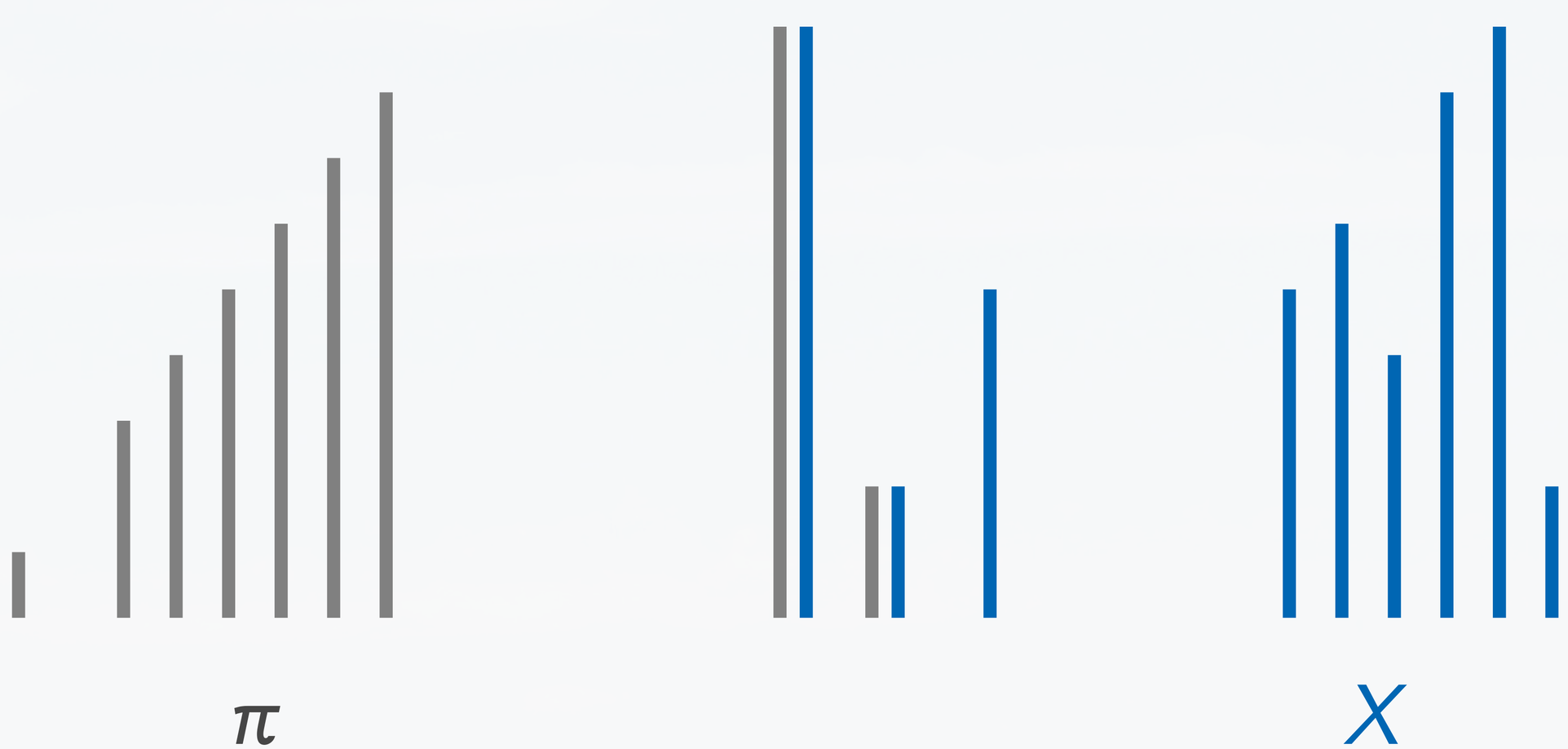

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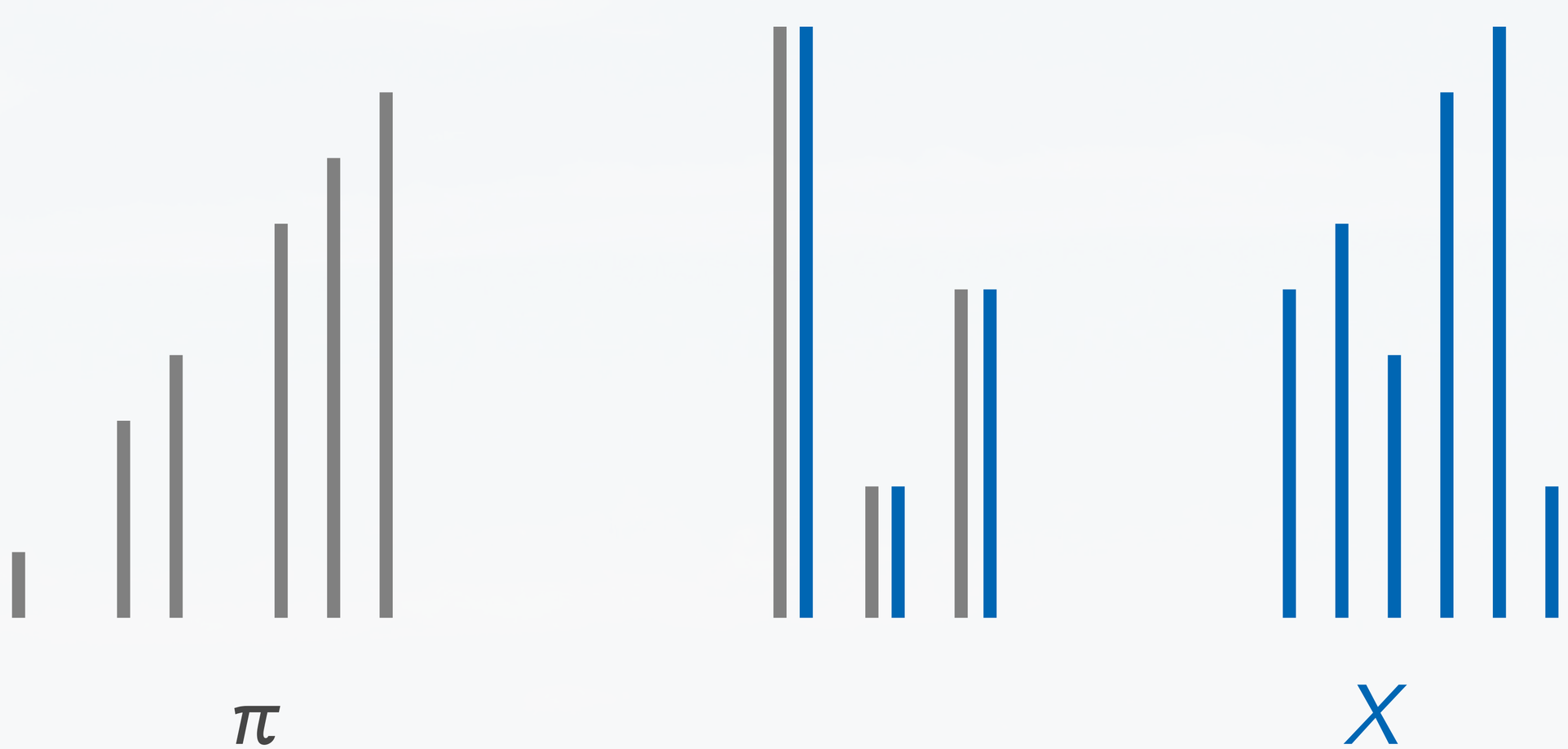

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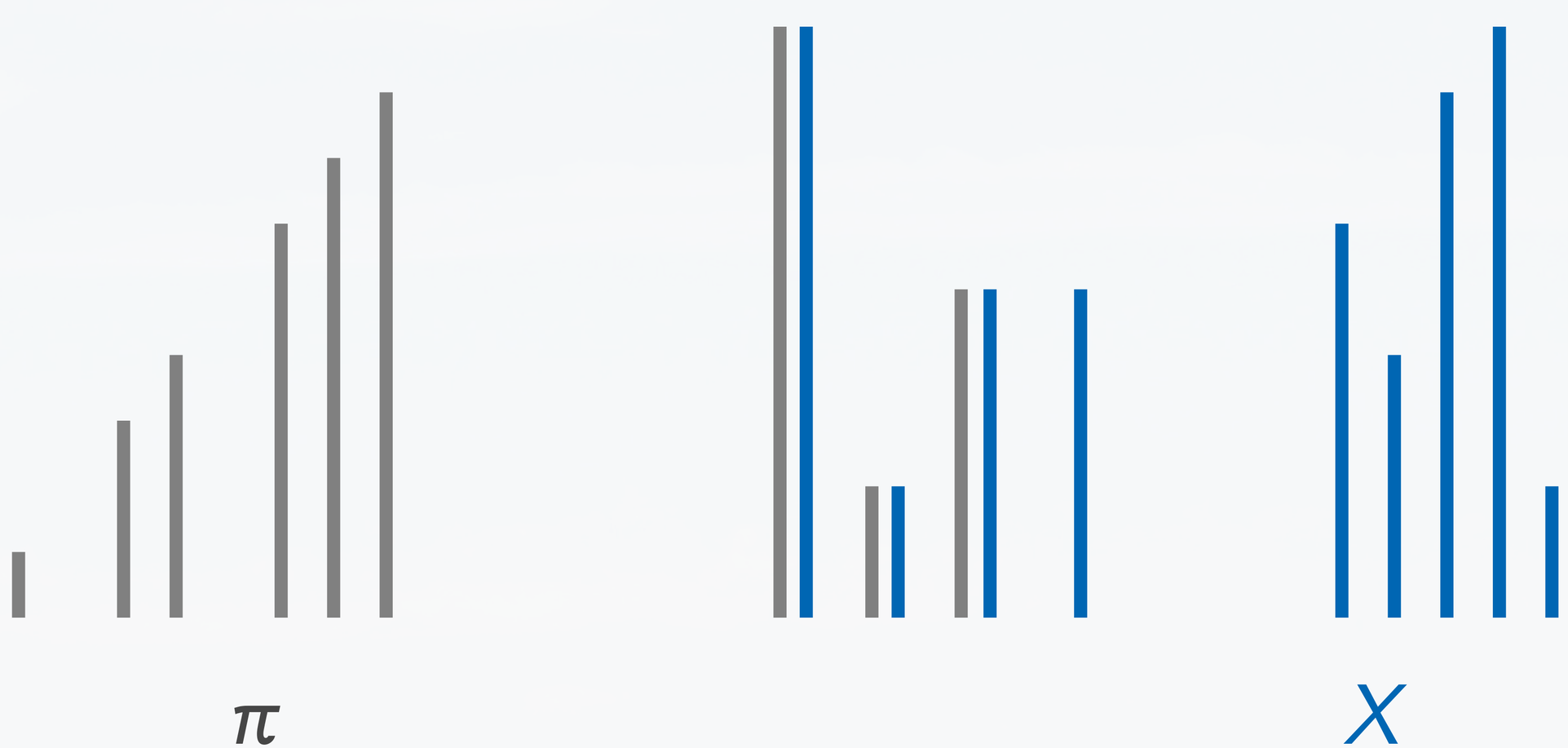

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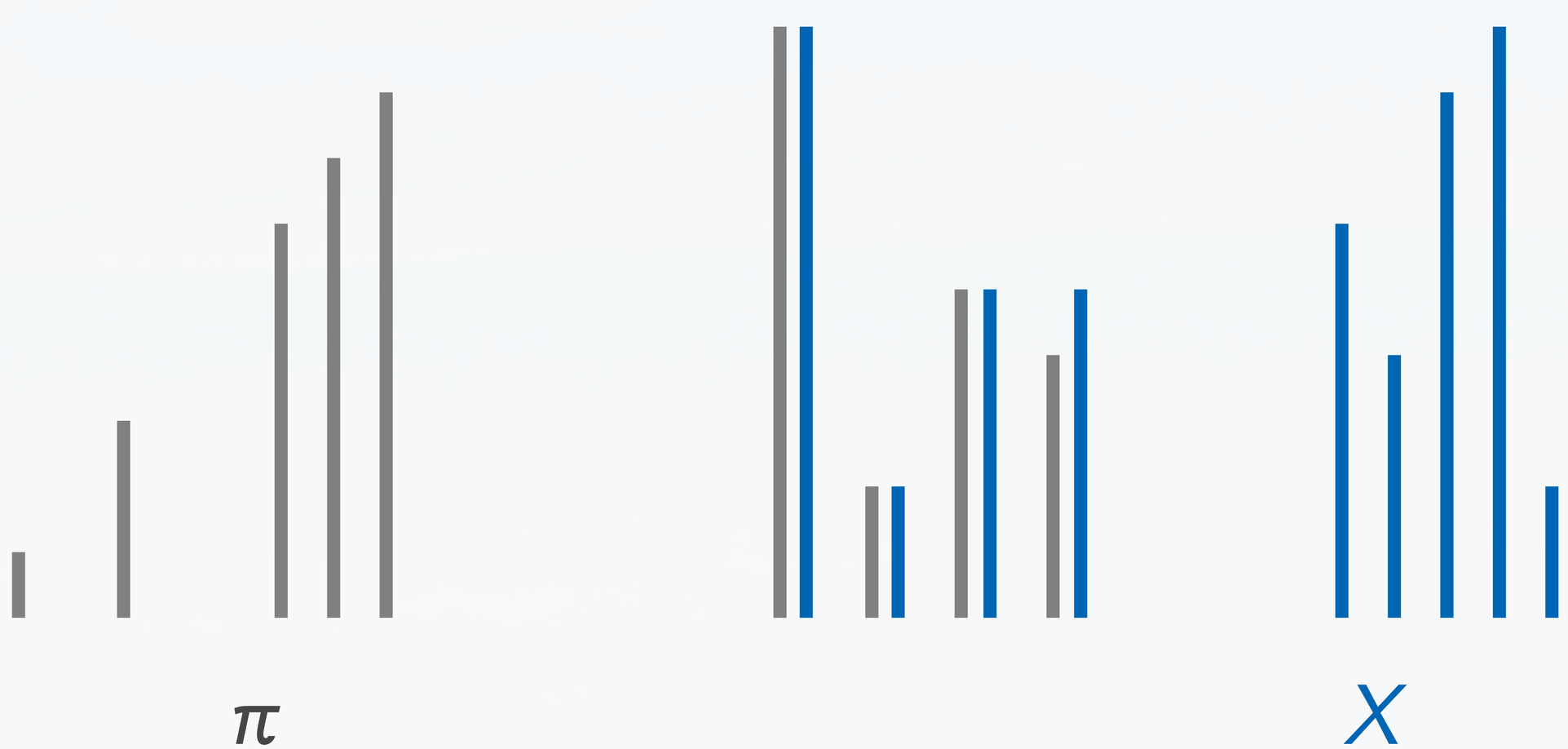

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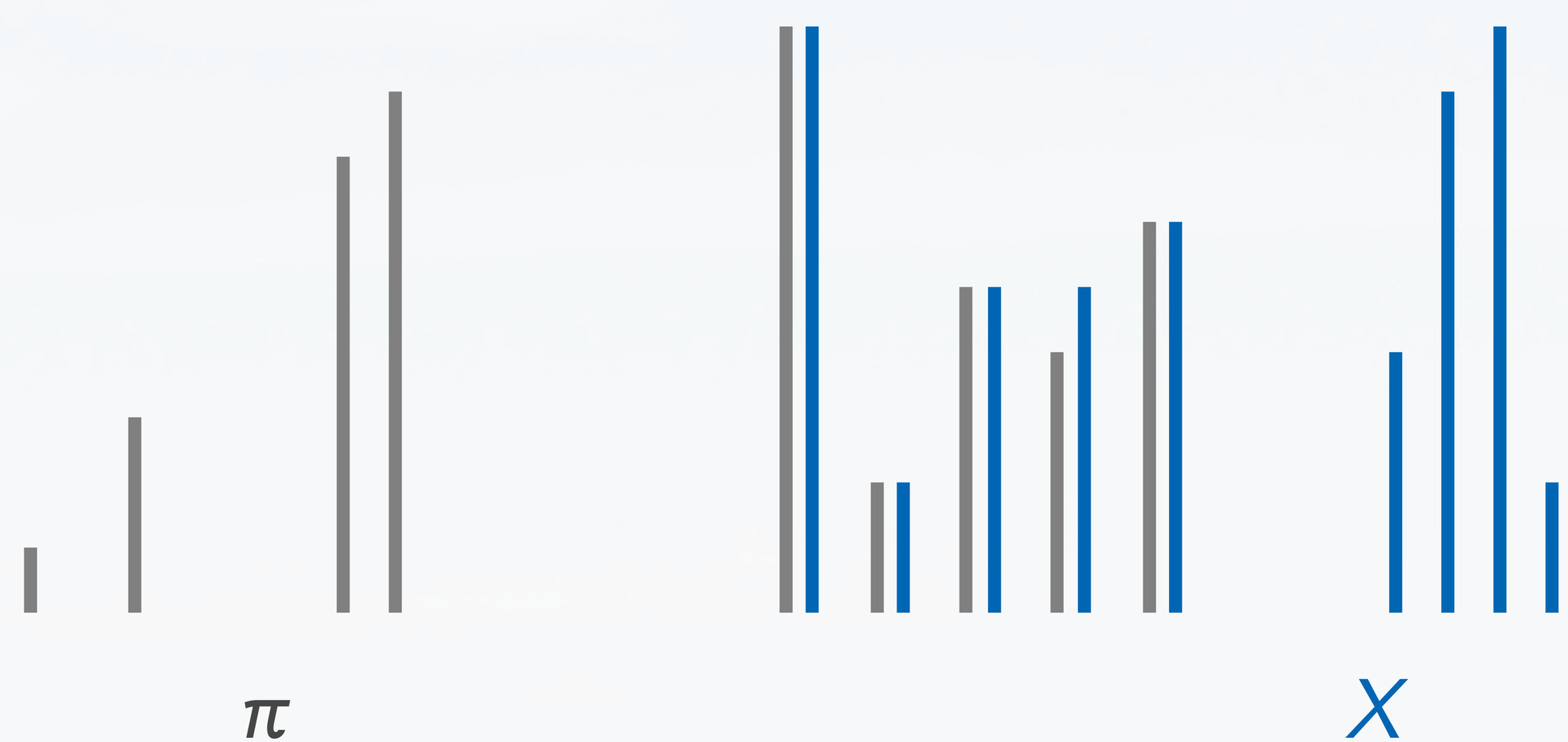

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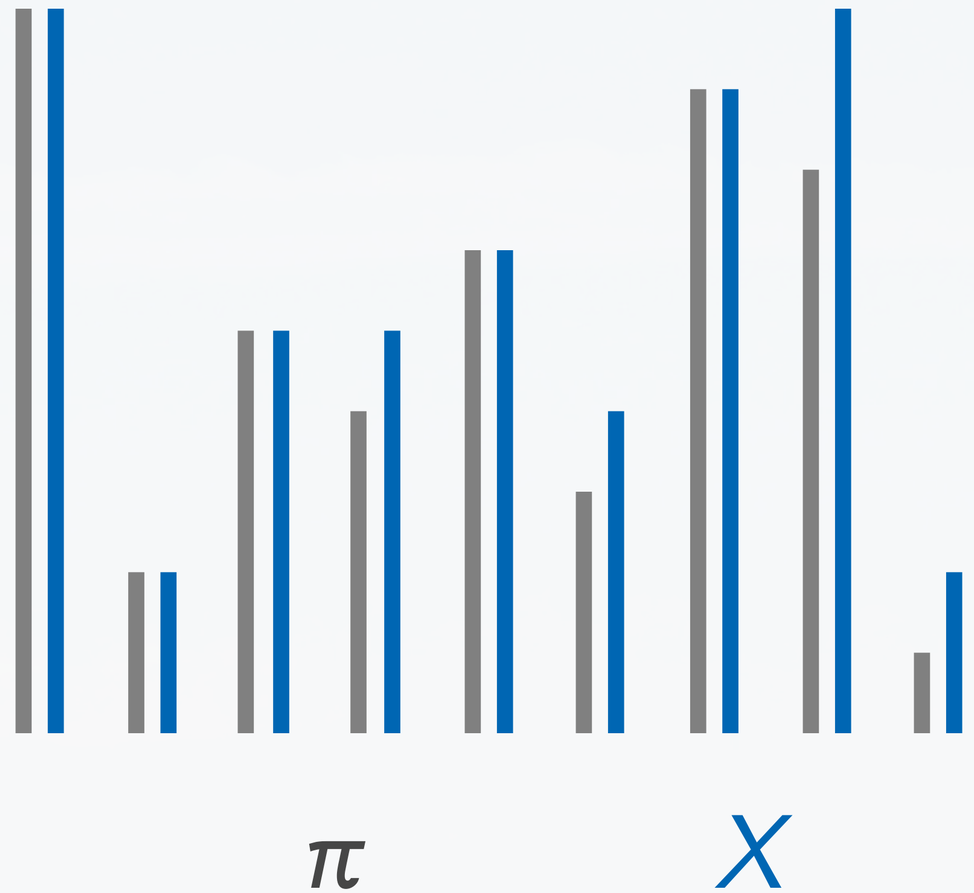

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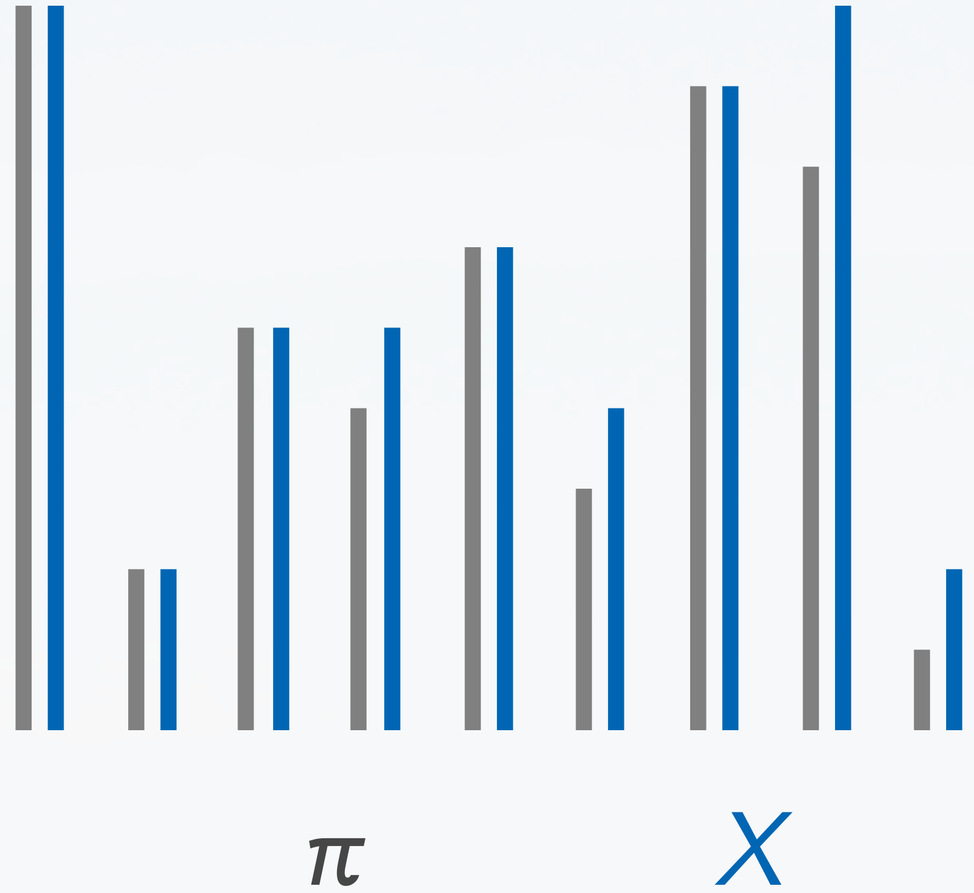
Implementation using union find in $\mathcal{O}(n \cdot \alpha(n))$.



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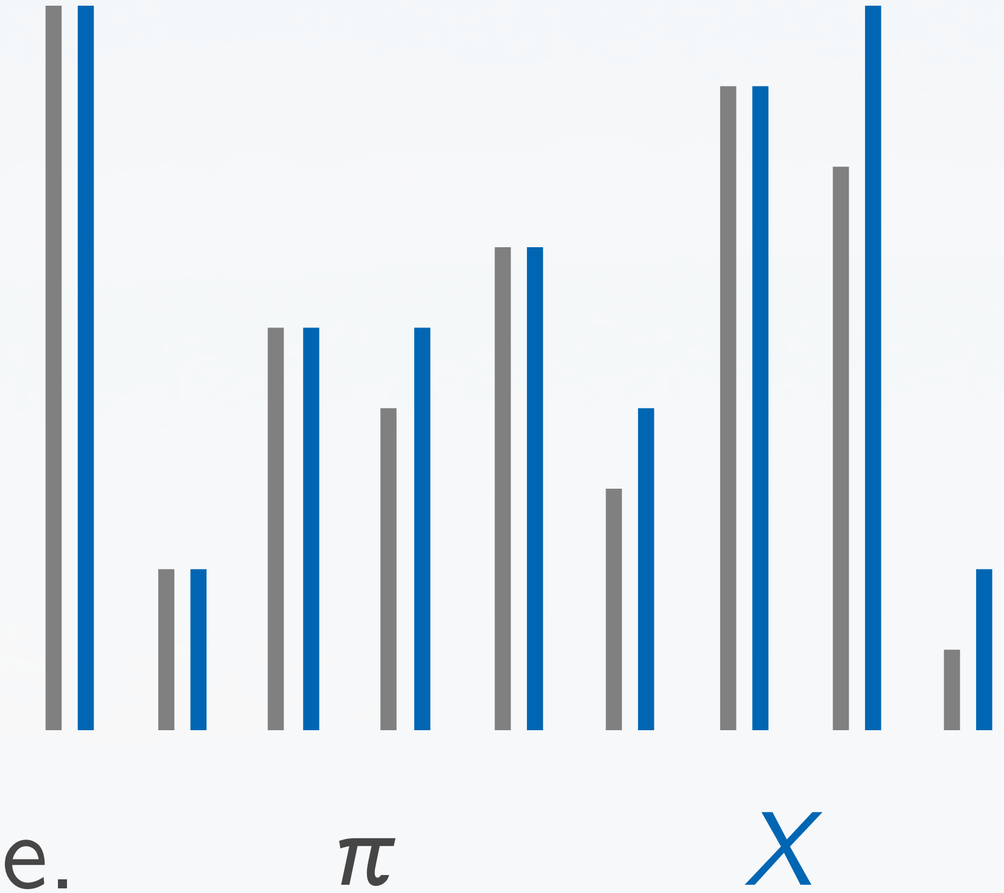
*Main advantage:*

Also works if the constraints arrive online.



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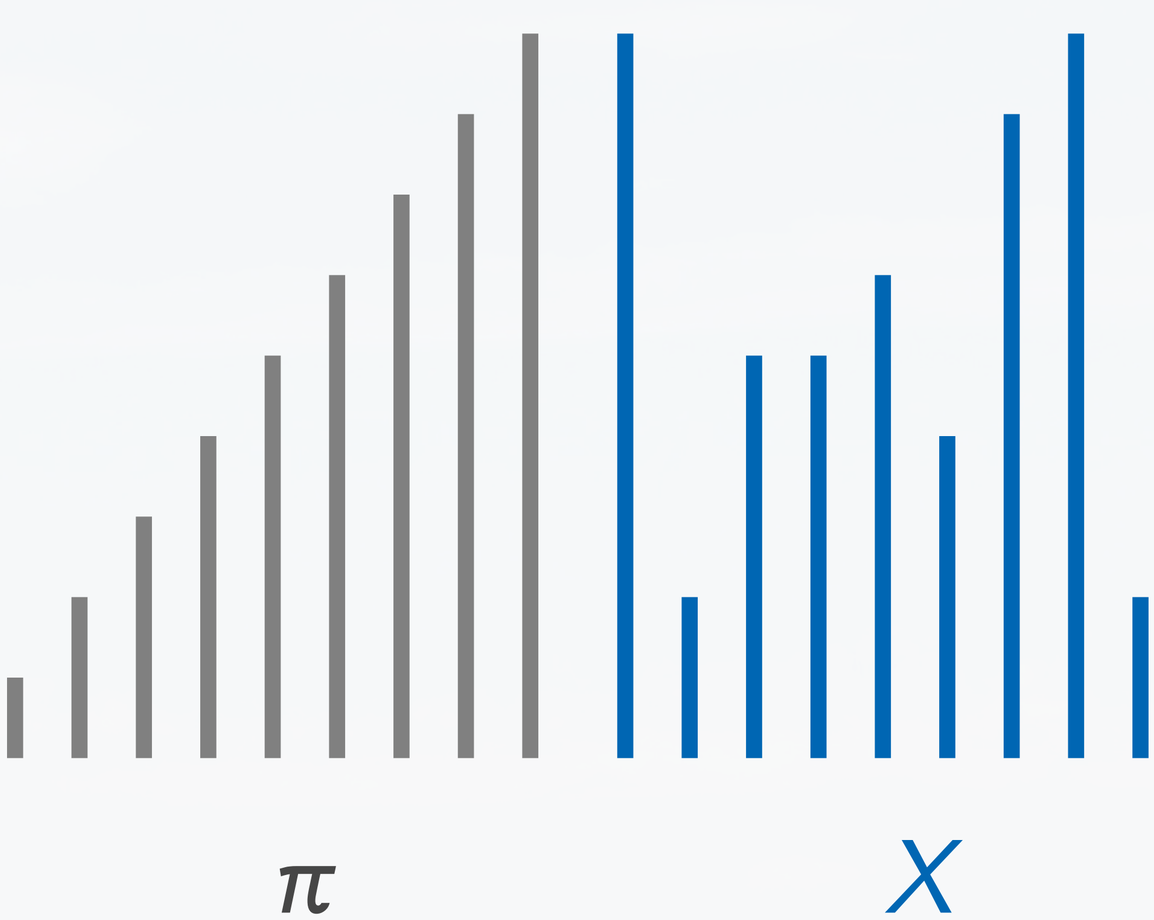
# Online Setting



How much information about the future is needed to fill the slots optimally?



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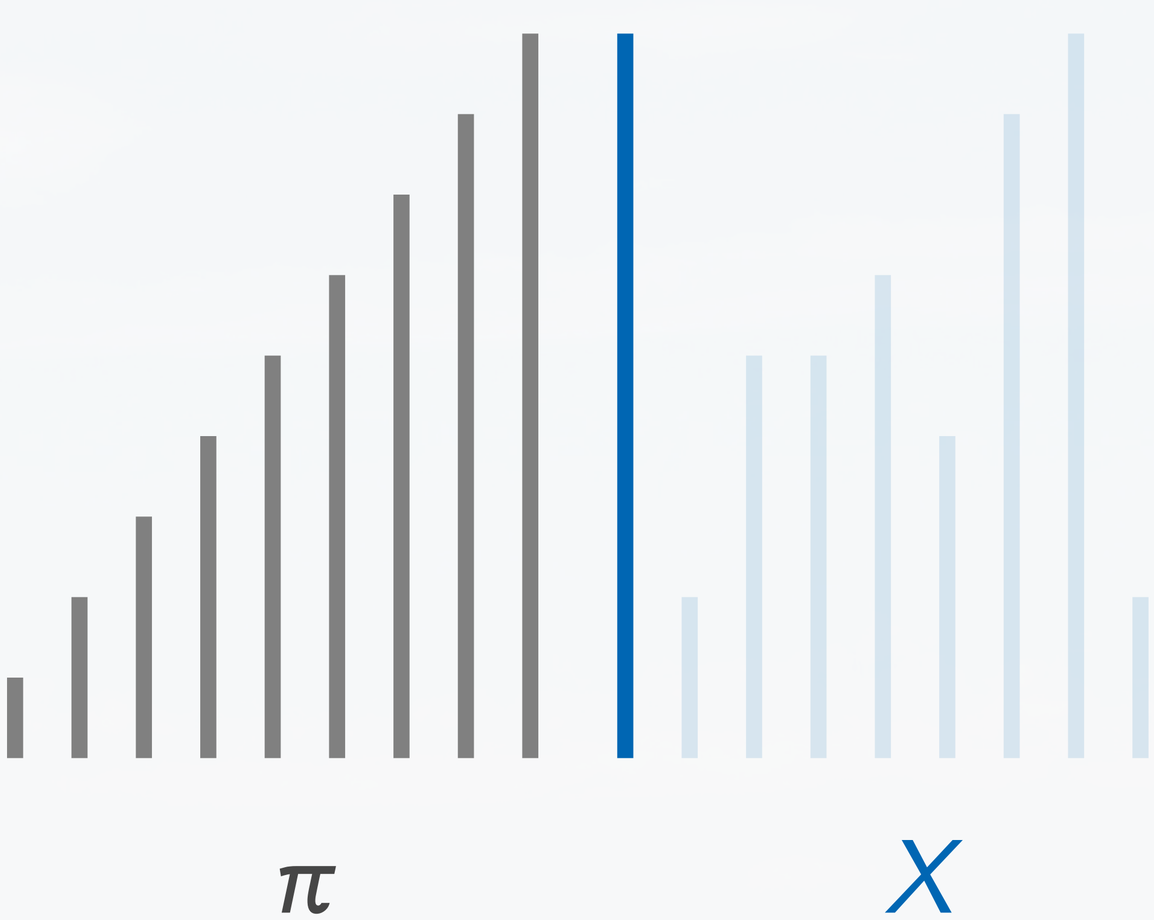

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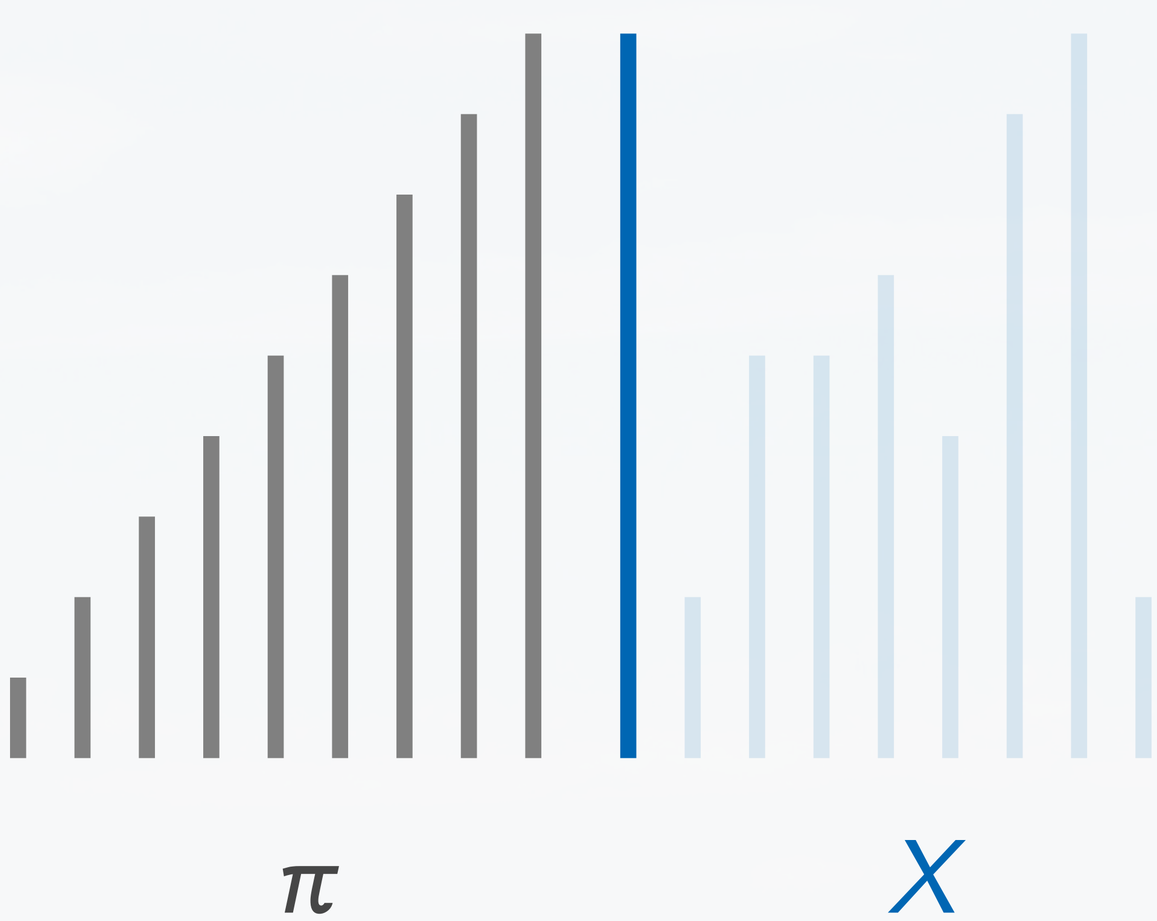



In our setting:

*No-swap model:*

$\rightarrow$ next 2 arrivals

*Swap model:*

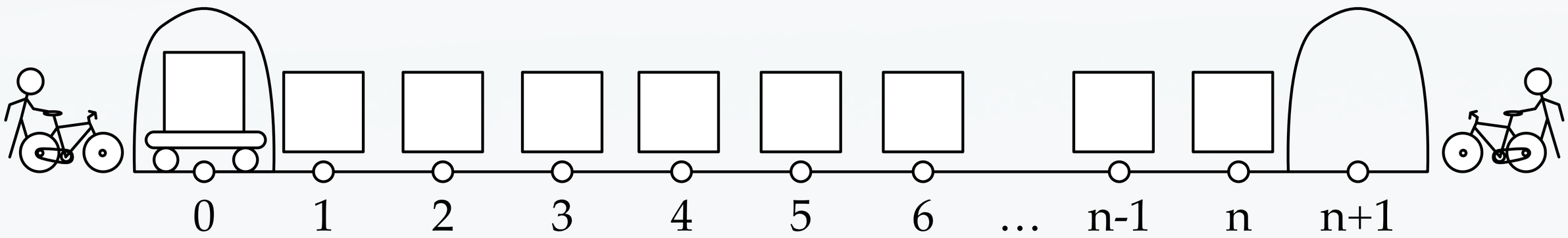
$\rightarrow$ only next arrival

# Two Doors



New setting:

- second door at the other end
- swap-robot, carrying a box initially
- still arrival-only and no rearrangement




Who picks the door for each customer?

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# Fixed Doors

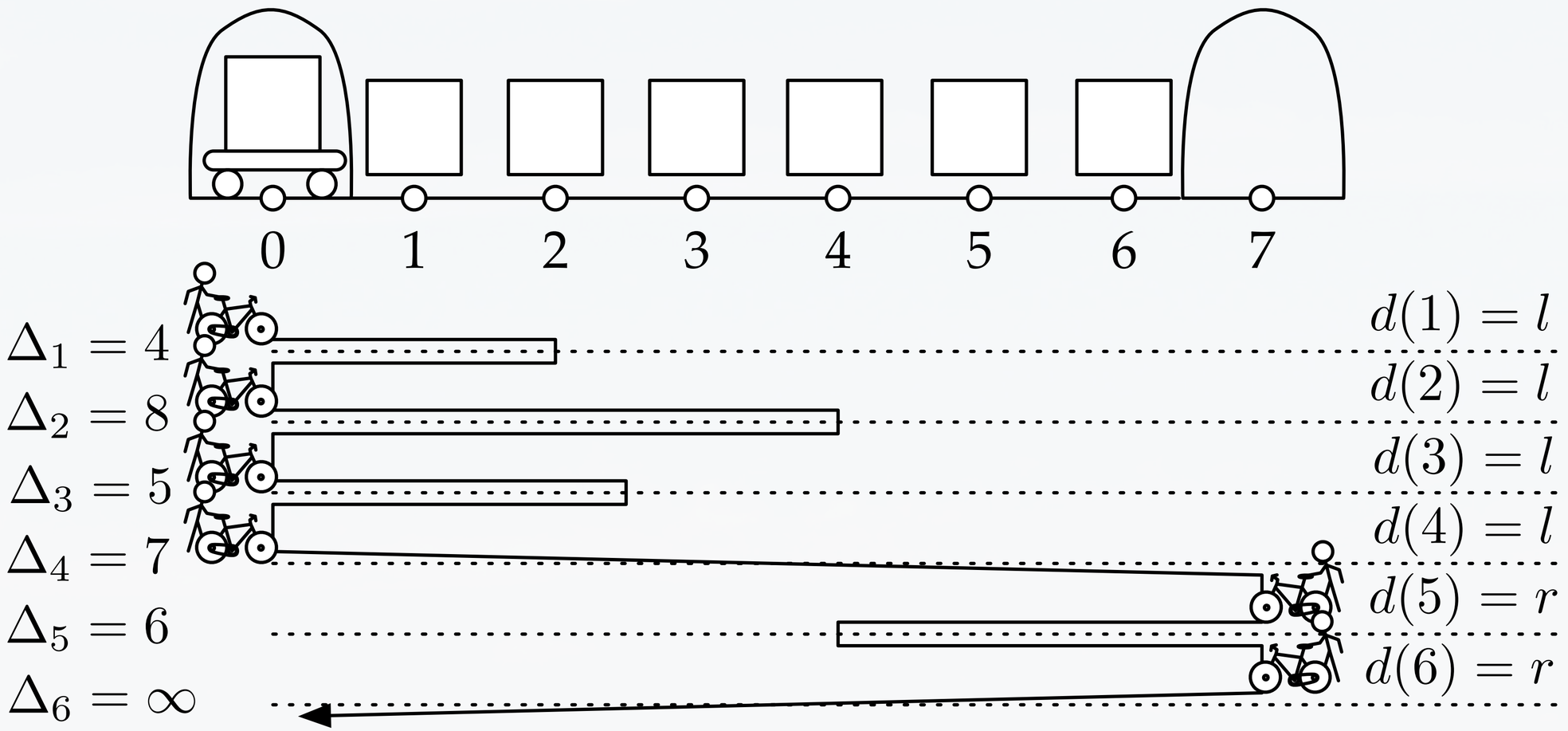


*Given:*

- inter arrival times $\Delta$
- door assignment $d$

*Wanted:*

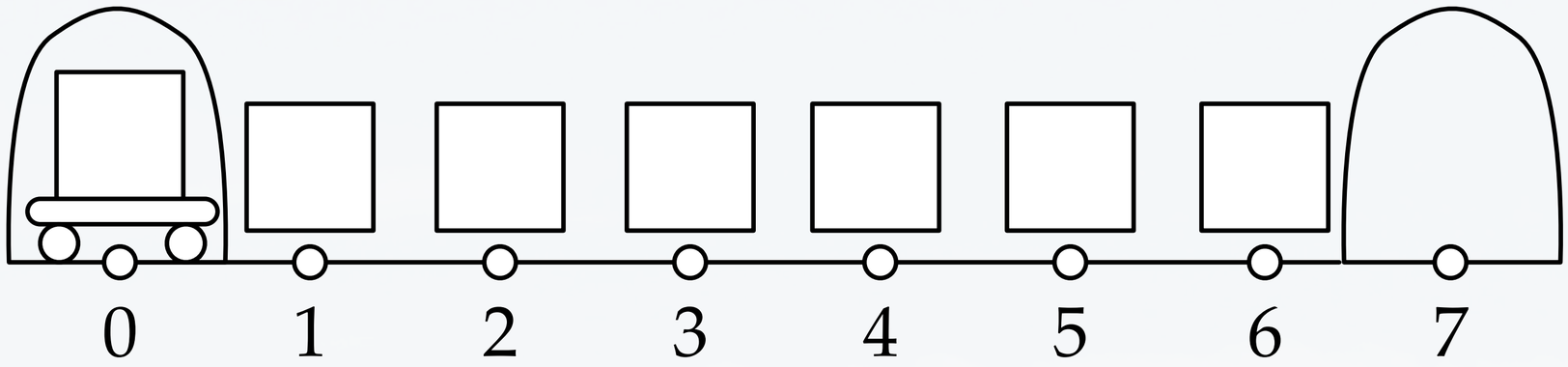
- slot assignment $\pi$



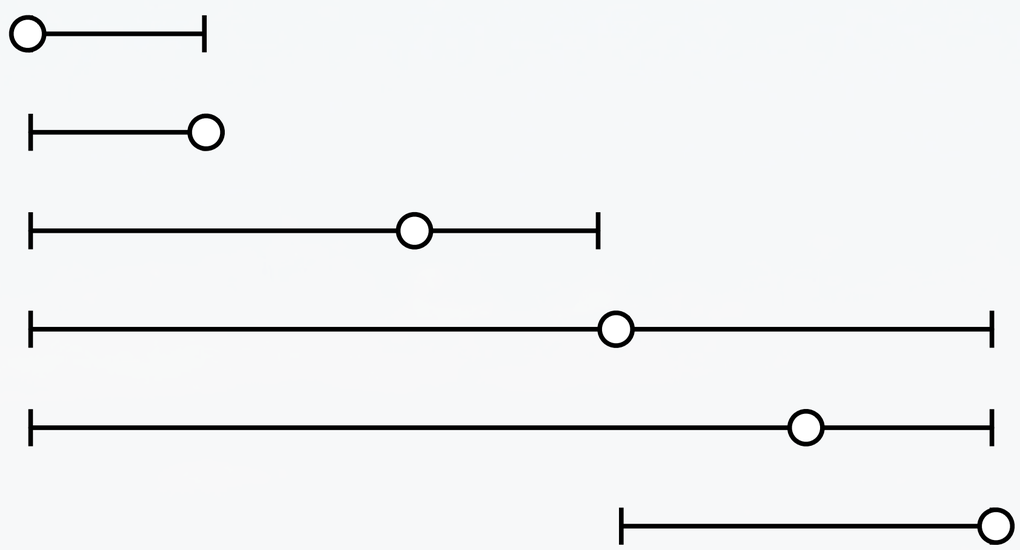
# Fixed Doors



*Solution:* earliest deadline first scheduling




$$\sigma = (1, 3, 2, 4, 6, 5)$$



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# Flexible Doors



*Given:*

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# Flexible Doors



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*Theorem:* For a given $\Delta$, it is *NP*-complete to decide whether feasible $\pi$ and $d$ exist.

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*Proof:* Reduction from *3SAT*.

# 3SAT



*Definition:* **3SAT**

- input: CNF-formula $F$ with
  - $M$ clauses $C_1, C_2, \ldots, C_M$ with $C_i = \{l_{i_1}, l_{i_2}, l_{i_3}\}$
  - $N$ variables $x_1, \ldots, x_N$
- output: can we satisfy all the clauses?

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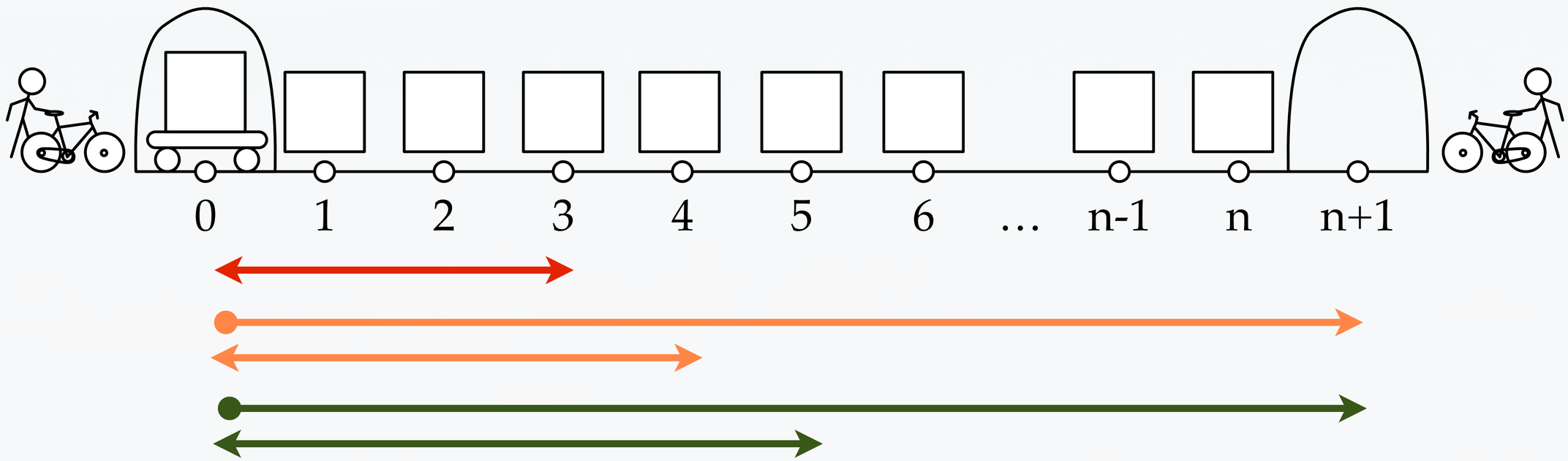
*Theorem [1971, Cook], [1973, Levin]:*
3SAT is *NP*-complete.

# Request Sizes



We classify inter request times $\Delta_i$ by duration:

- short: $\Delta_i \leq n$: the other door is unreachable
- medium: $n < \Delta_i < 2n$: in between
- long: $2n < \Delta_i$: switch or not and reach any slot



# Request Blocks



Consecutive short requests have to be assigned to the same door *en bloc*.

Transform a given *3SAT* formula into a sequence of blocks and interleave them with long requests.

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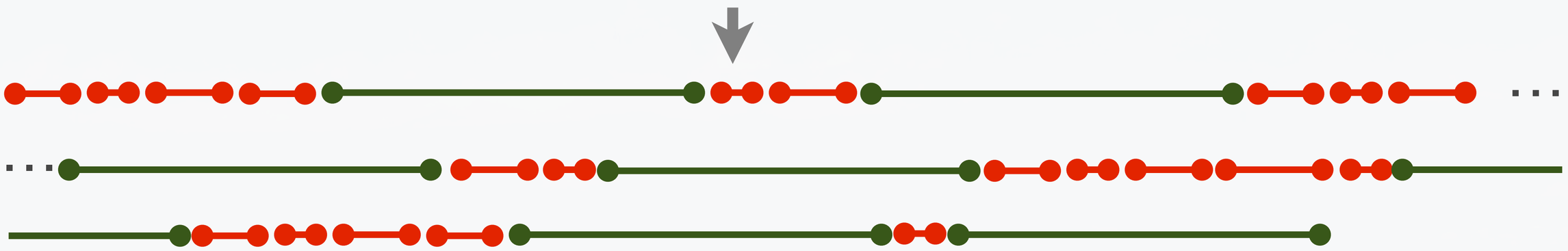
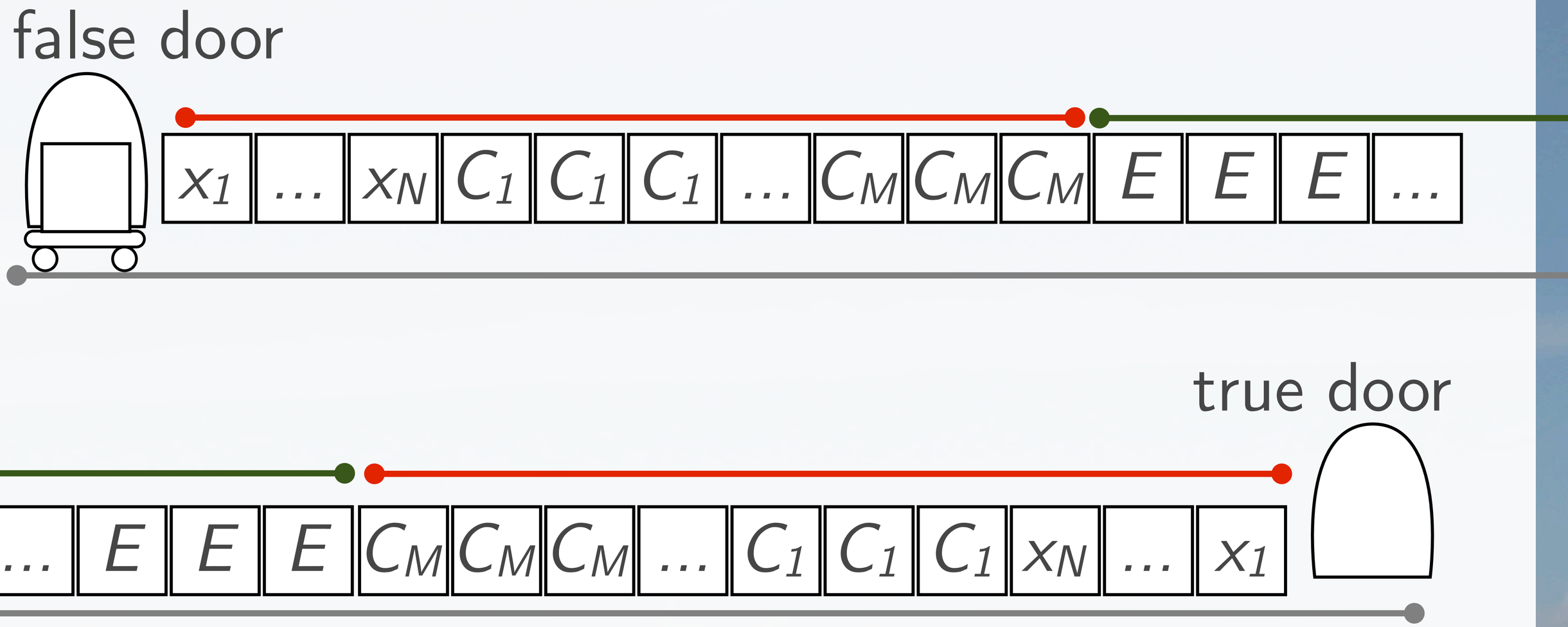


# Layout

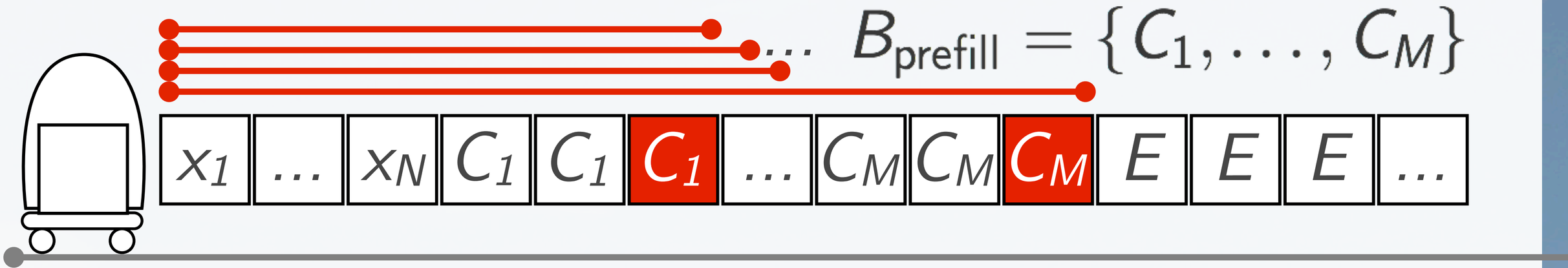



false door
$x_1$ ... $x_N$ $C_1$ $C_1$ $C_1$ ... $C_M$ $C_M$ $C_M$ $E$ $E$ $E$ ...
... $E$ $E$ $E$ $C_M$ $C_M$ $C_M$ ... $C_1$ $C_1$ $C_1$ $x_N$ ... $x_1$
true door

# Blocks



**Prefill block:** fill one slot per clause on the false side



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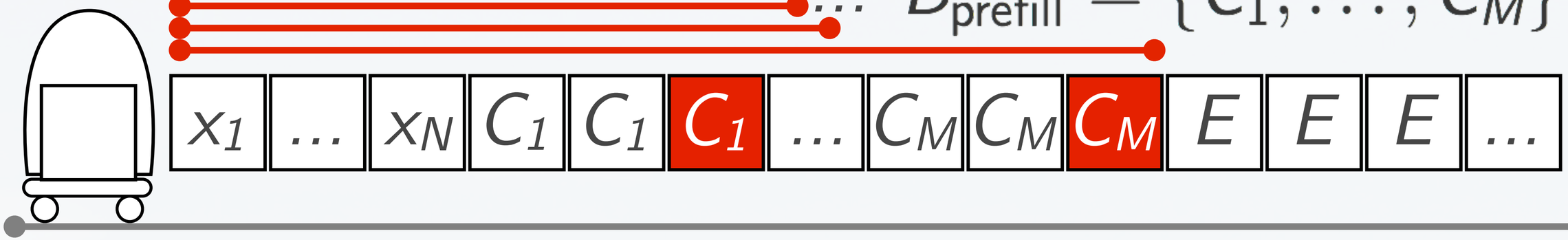
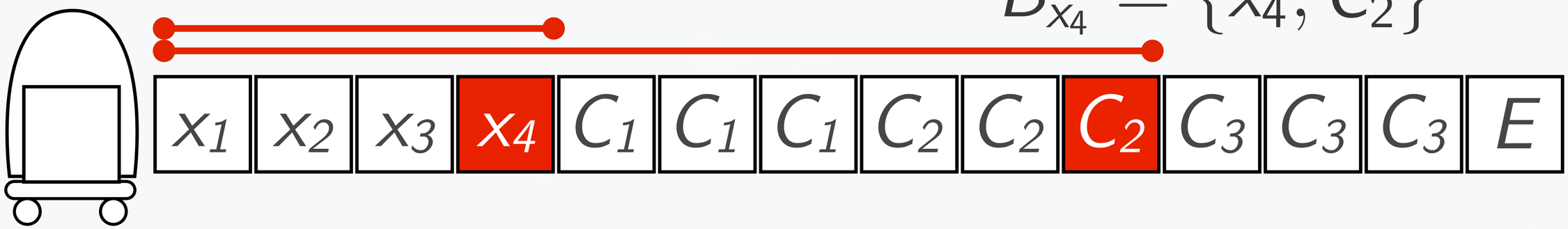
$B_{\text{prefill}} = \{C_1, \ldots, C_M\}$




**Literal blocks:** variable and all clauses containing it

*Example:* $F = \{\{x_1, x_3, \bar{x}_4\}, \{x_2, \bar{x}_3, x_4\}, \{\bar{x}_1, x_2, \bar{x}_4\}\}$

$B_{x_4} = \{x_4, C_2\}$



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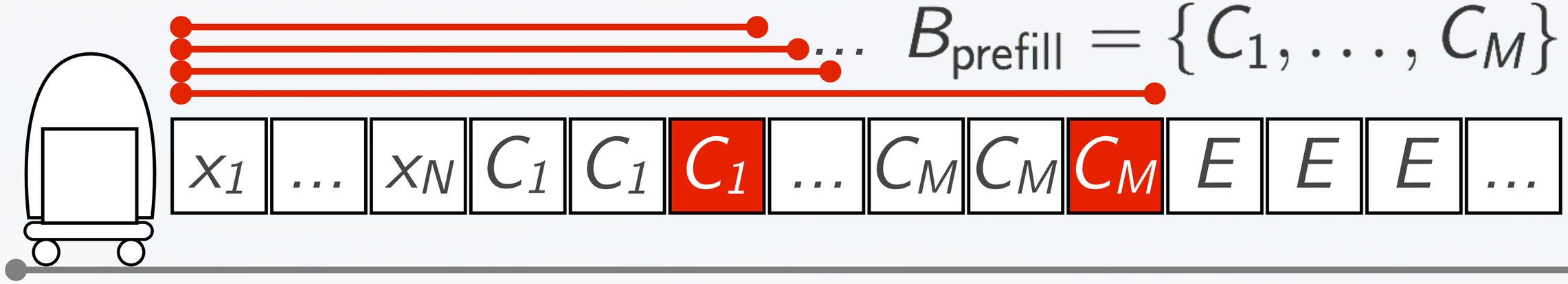


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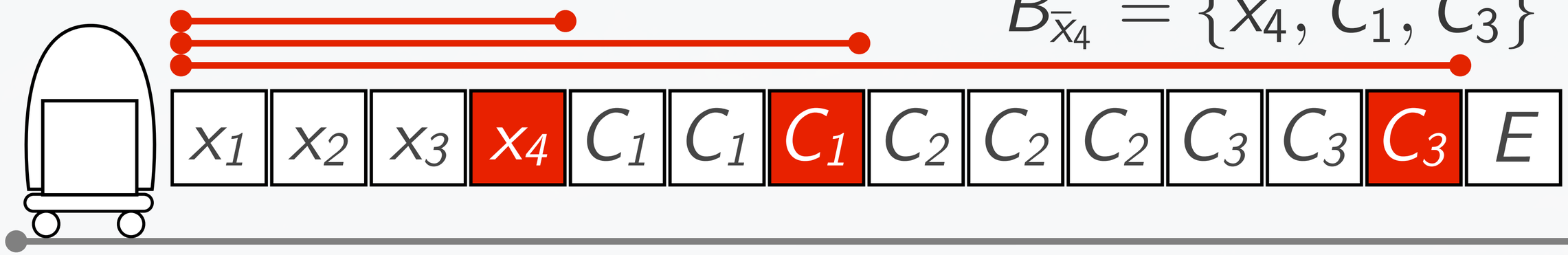
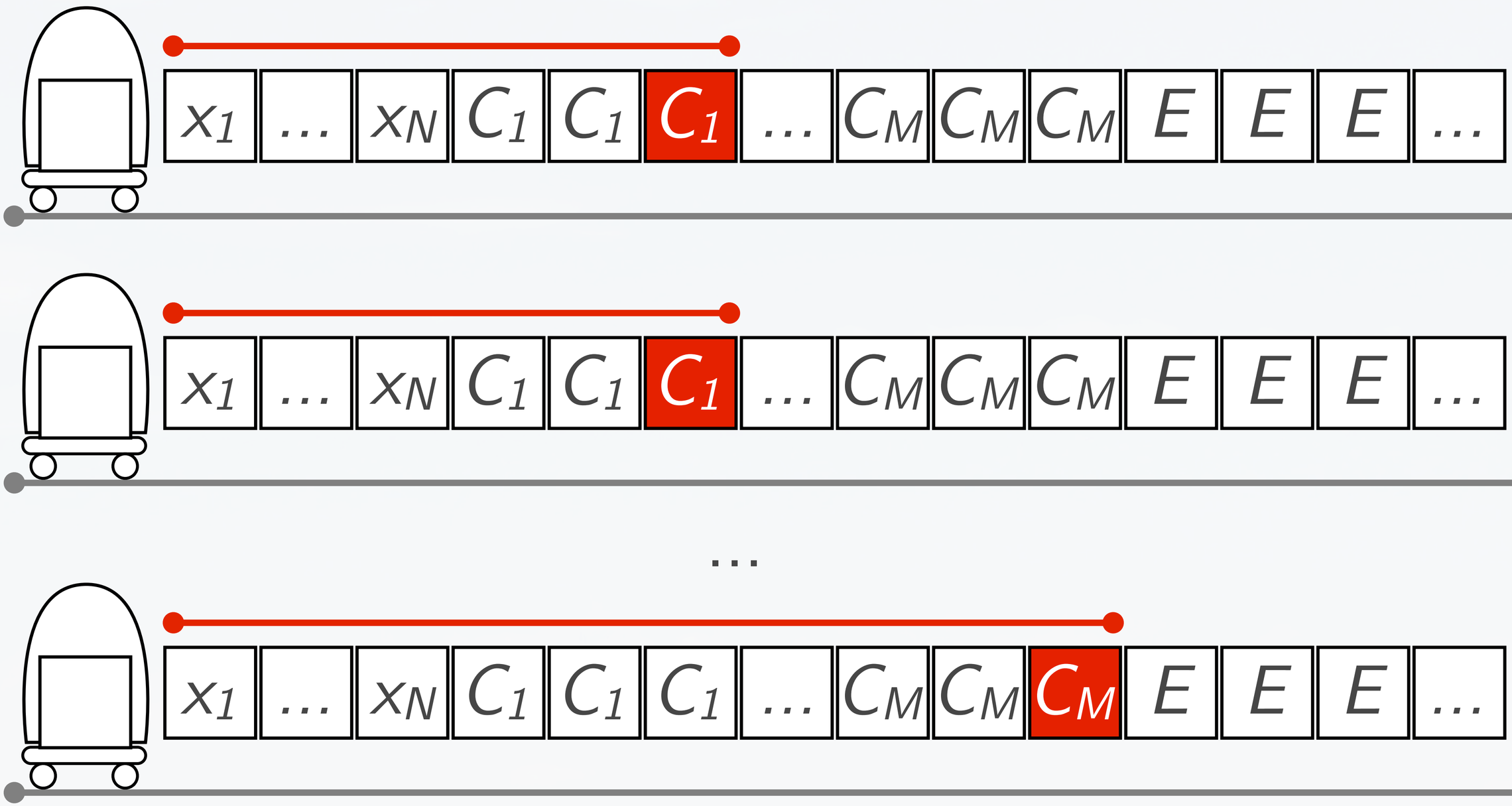


# Blocks



**Fillup blocks:** two single request blocks per clause



# Proof



*Claim:* The two door problem has a solution if and only if the *3SAT* formula $F$ is satisfiable.

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*Proof:*

- Per label there are as many slots as requests.
- The long requests fill up the $E$ slots.
- Opposite literals get assigned to opposite doors.
- Per clause at least one literal block has to get assigned to the true door as one slot on the false side is already taken by the prefill block.

# Entire Day



Back to a single door, but also with departures:

# Entire Day



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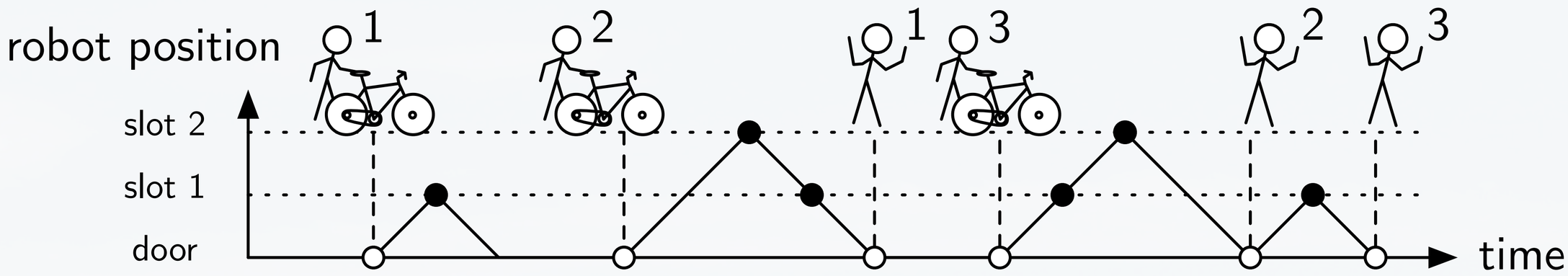
*Example:*



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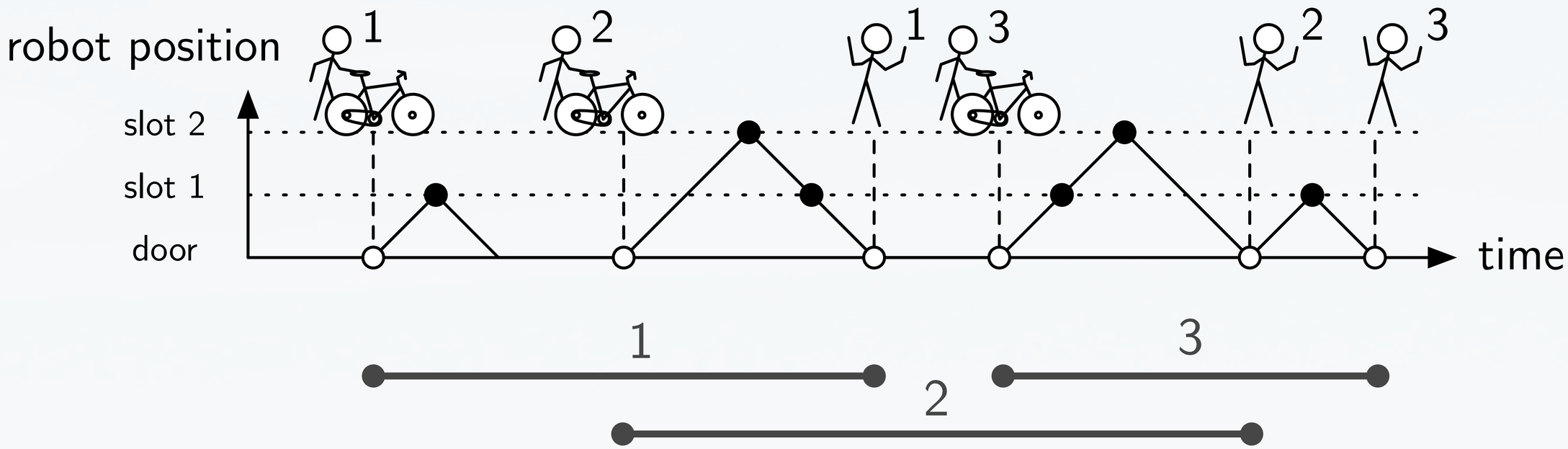
Back to a single door, but also with departures:

*Example:*

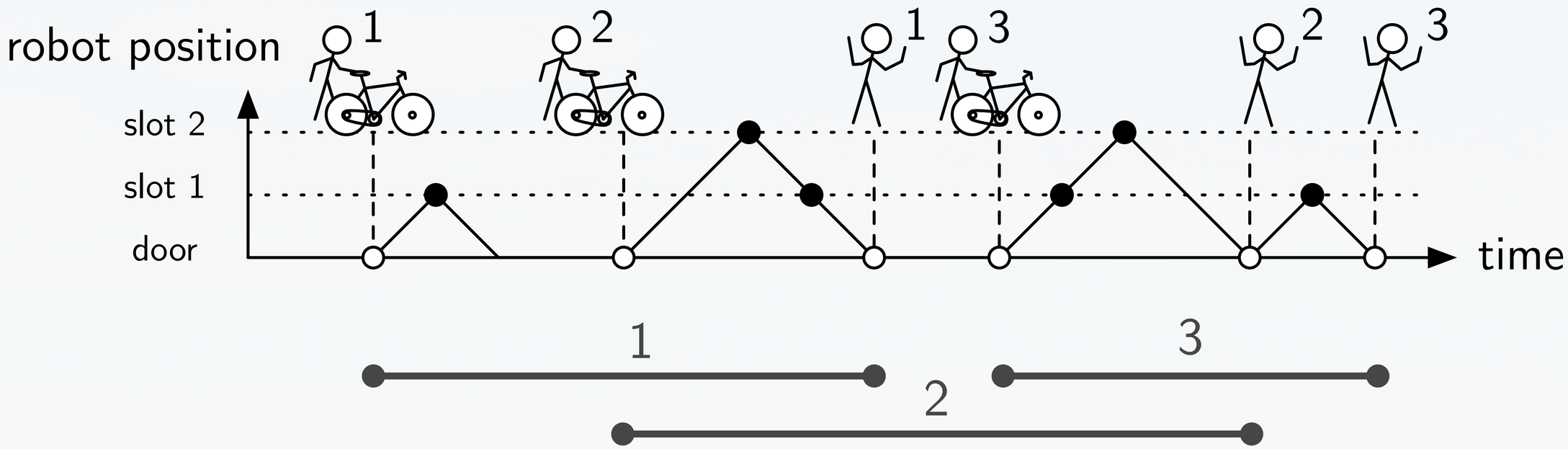


# Entire Day



Back to a single door, but also with departures:

*Example:* It is not just coloring intervals!



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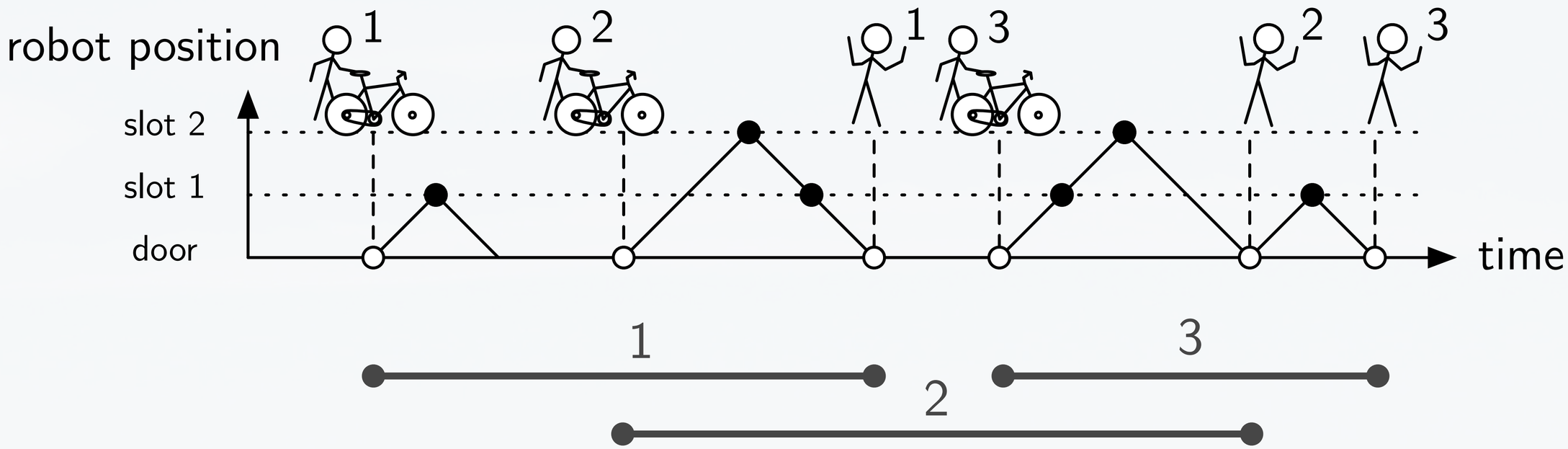


*Theorem:* It is *NP*-complete to find a feasible $\pi$.

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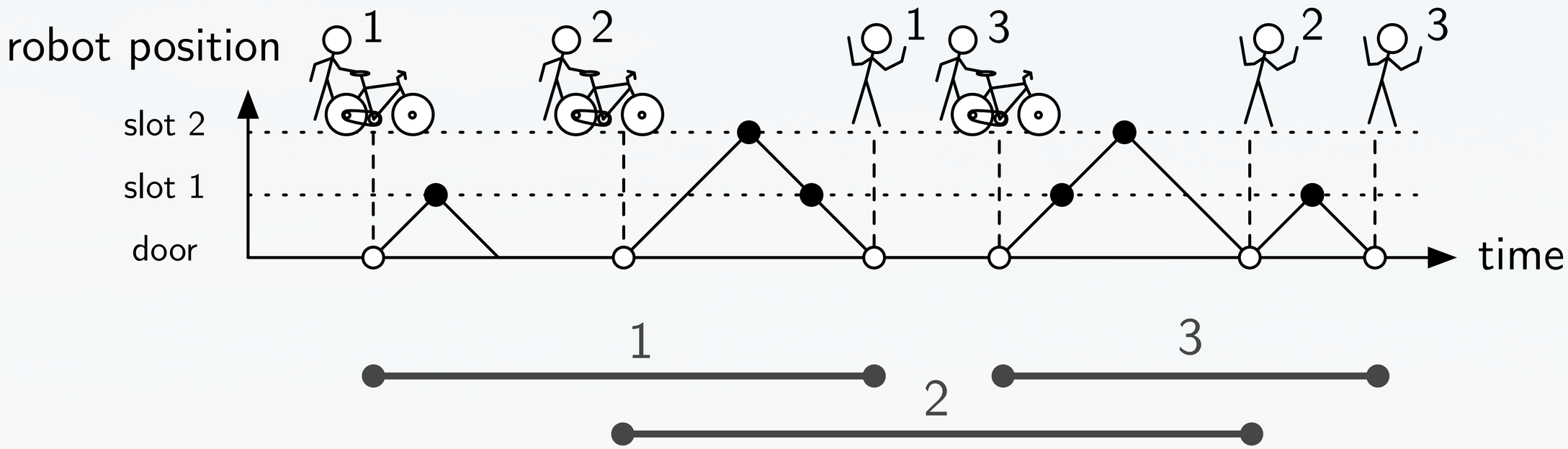


*Theorem:* It is *NP*-complete to find a feasible $\pi$.

*Proof:* Reduction from *Circular Arc Coloring*.

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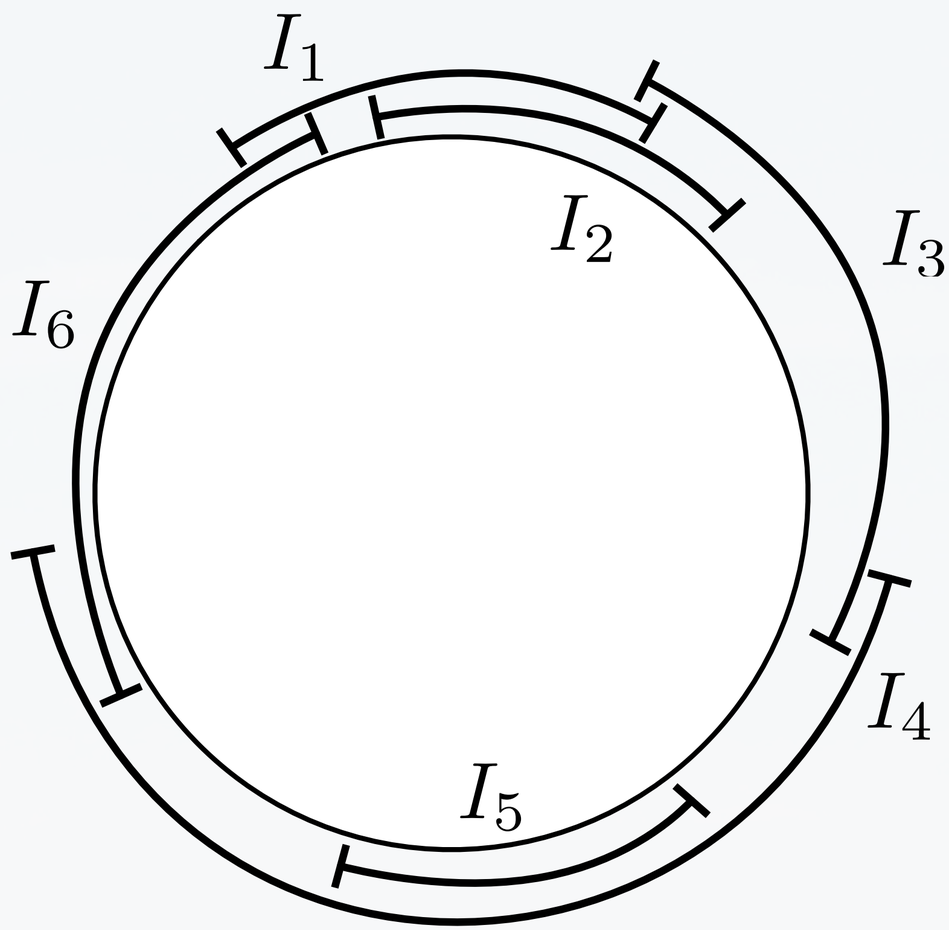
# Arc Coloring



*Definition:* **Circular Arc Coloring**

Given $I$ arcs on a circle, is there a $k$-coloring of their intersection graph?

*I=6, k=3*



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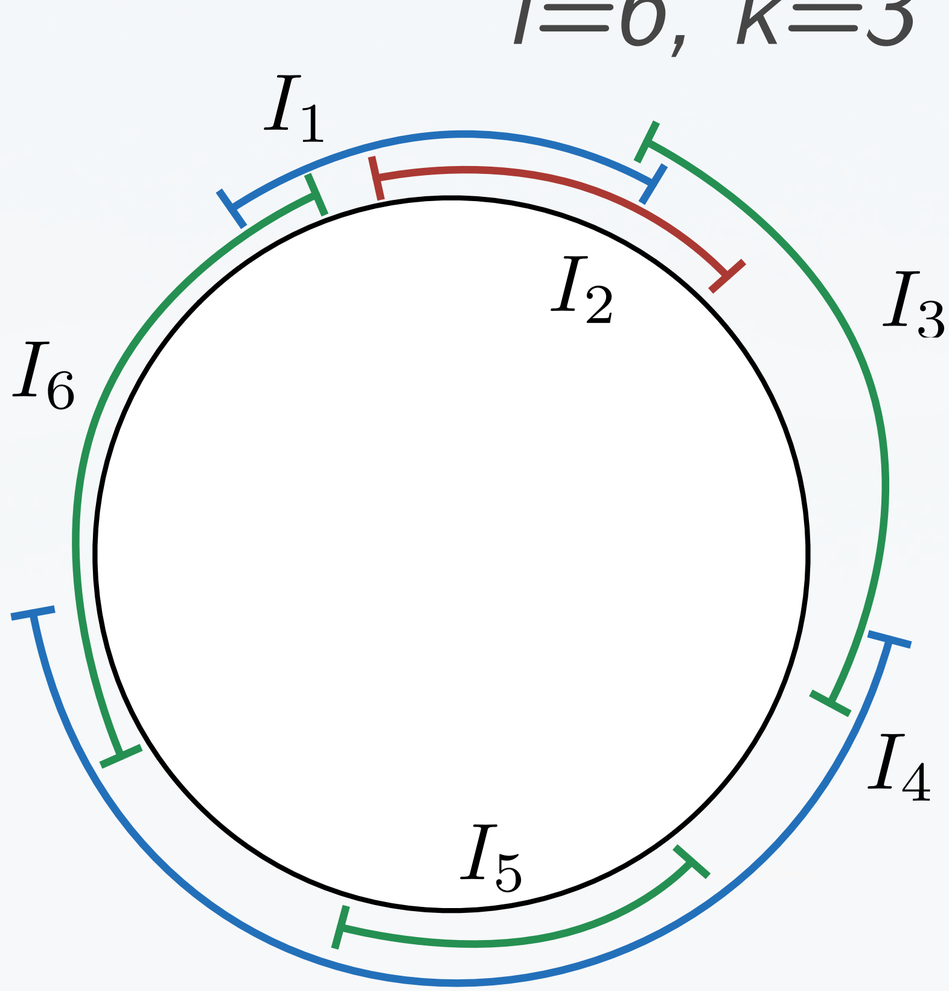

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# Arc Coloring



*Definition:* **Circular Arc Coloring**

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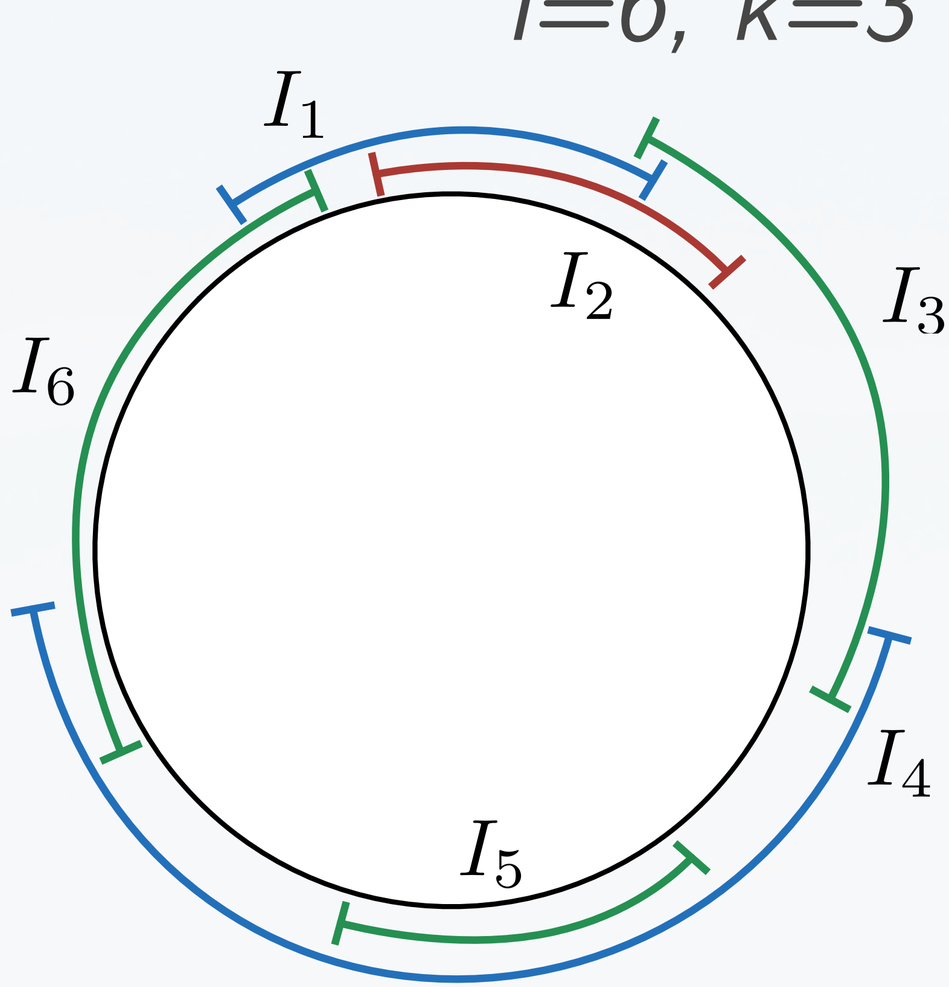
*Theorem [1980, Garey et al.]:*
Circular Arc Coloring is *NP*-complete.



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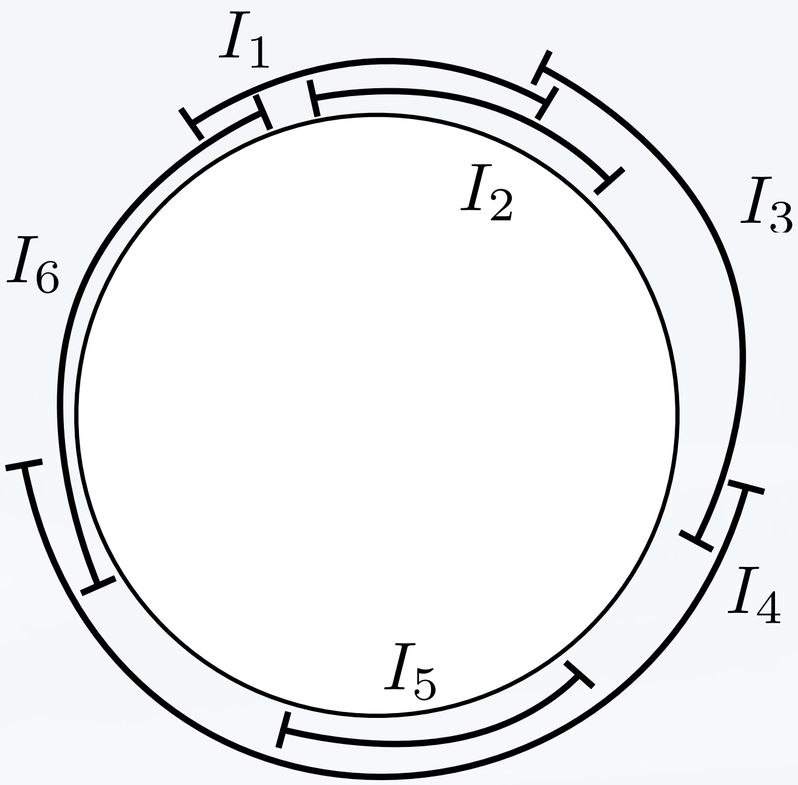
# Reduction






Outline of the reduction

- split the arcs at position 0
- arbitrarily, but consistently color the arcs across this cut
- transform it into an all-day instance with $k$ slots

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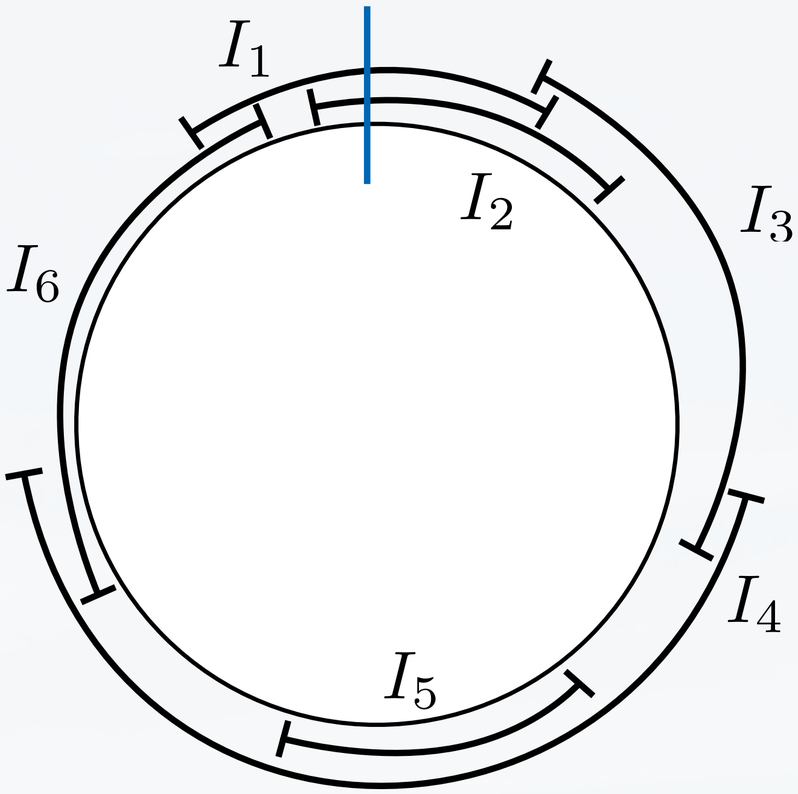
# Reduction



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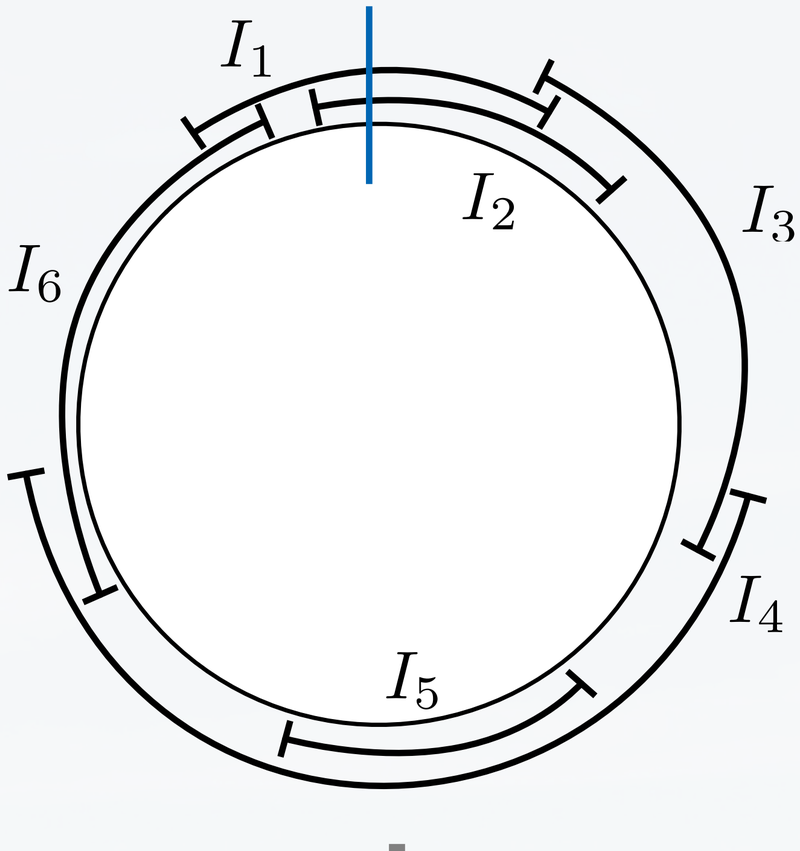


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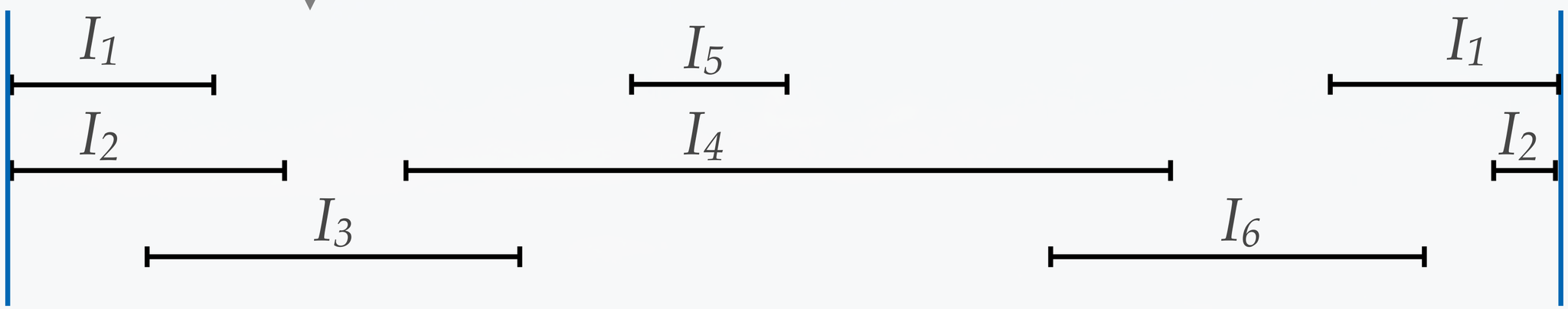

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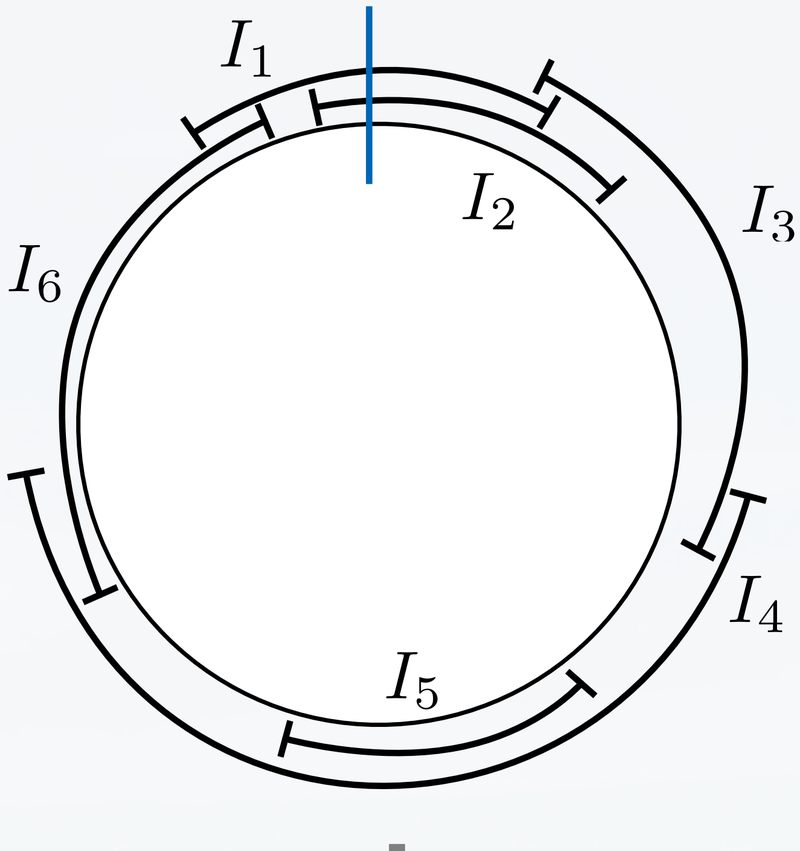
# Reduction



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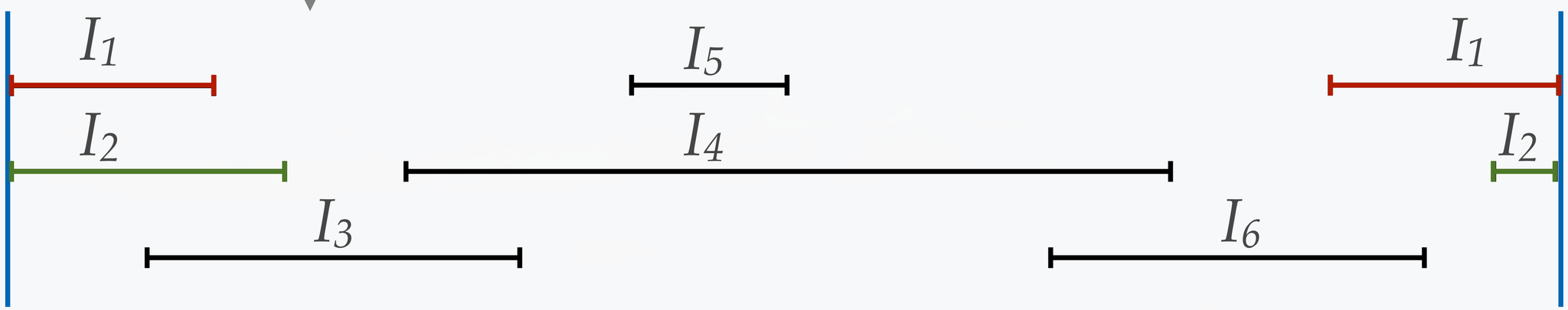

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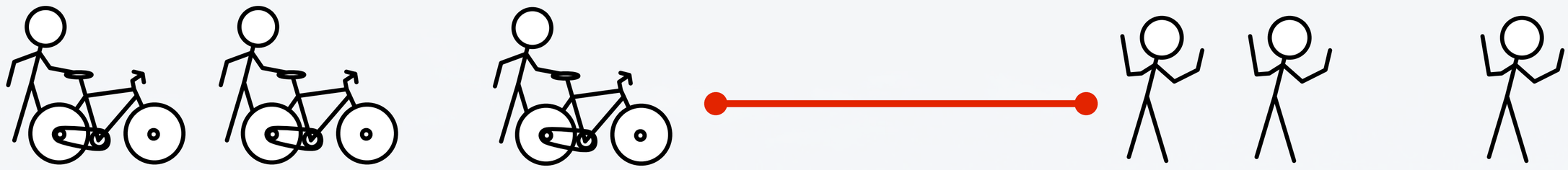
# Lunch Break





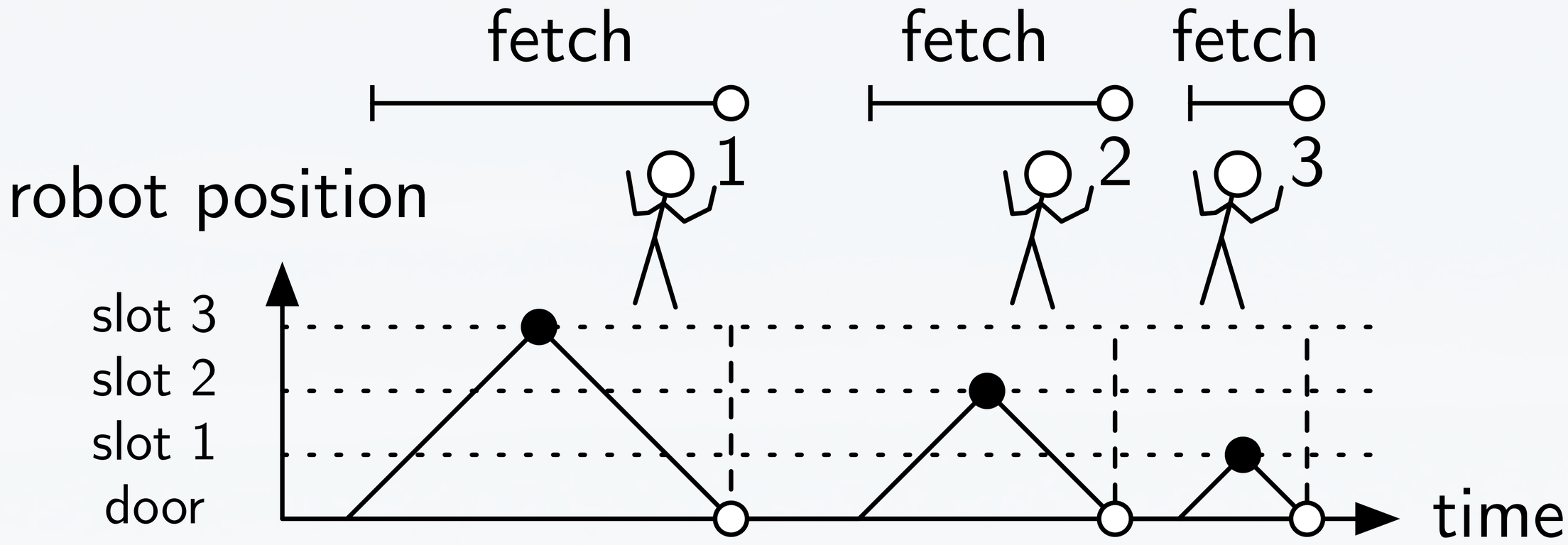
What if all the customers arrive in the morning and depart only in the evening?

- Every slot used only for a single customer
- Arrival-only problem in the morning
- Sorting problem during lunchtime
- Departure-only problem in the afternoon

# Afternoon



*Example:*




Solving another instance of the constrained permutation problem.

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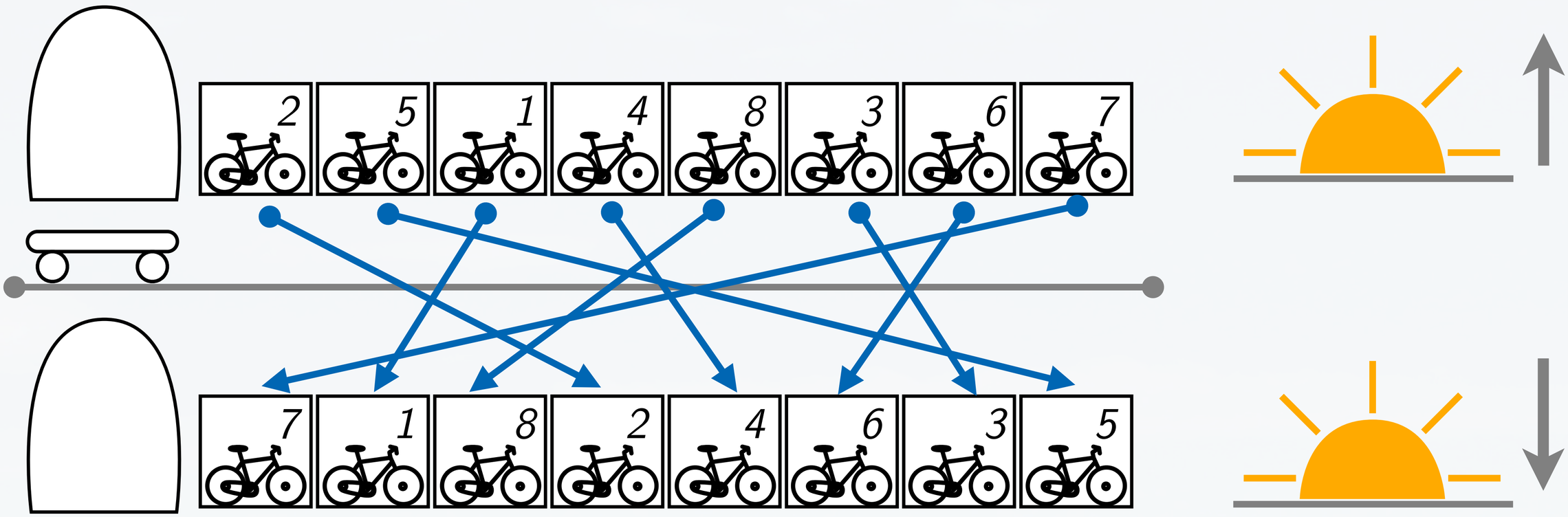
# Lunch Break










How do we sort this as quickly as possible?

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# Sorting Problem

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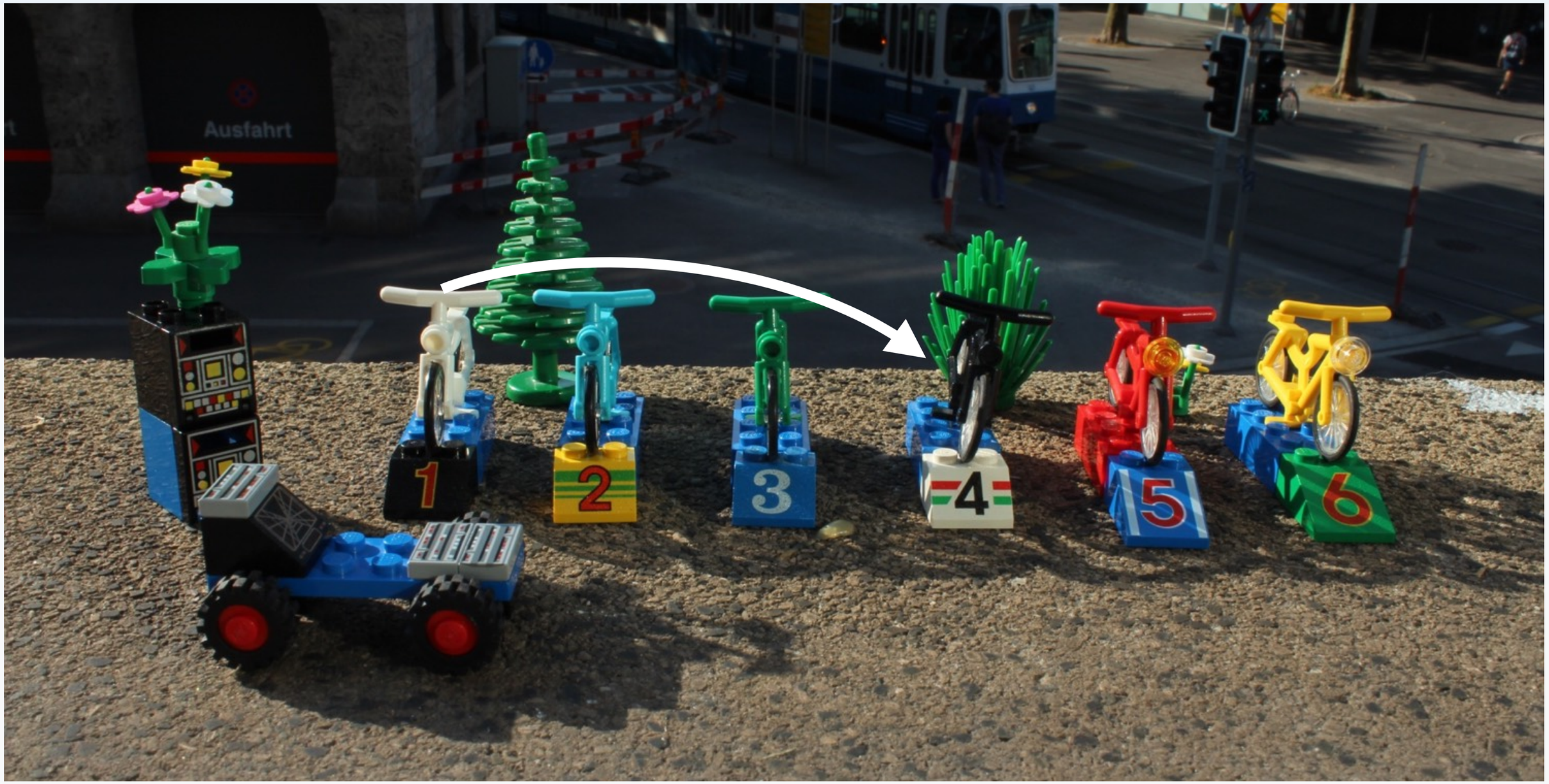
# Sorting Problem



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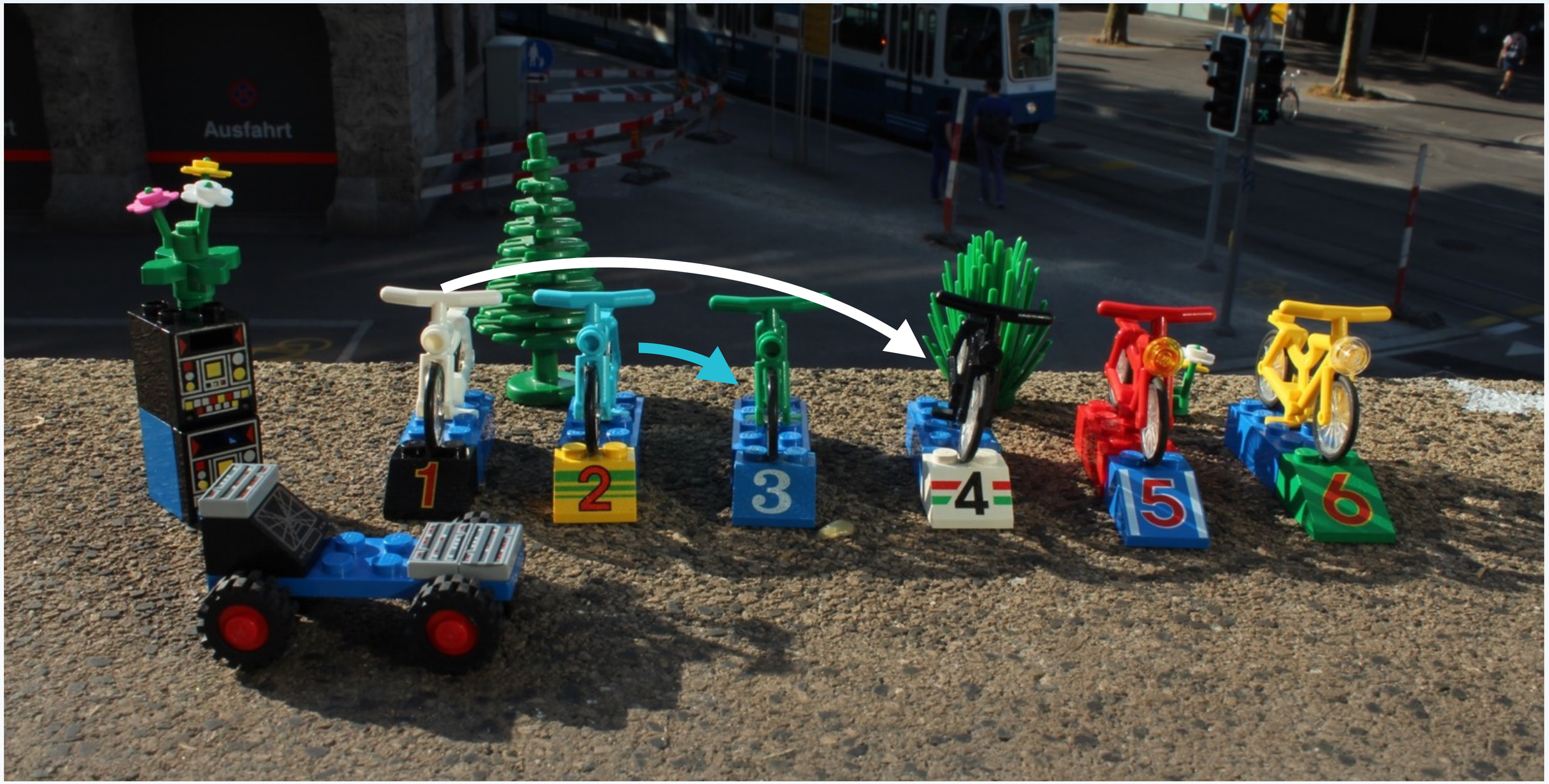

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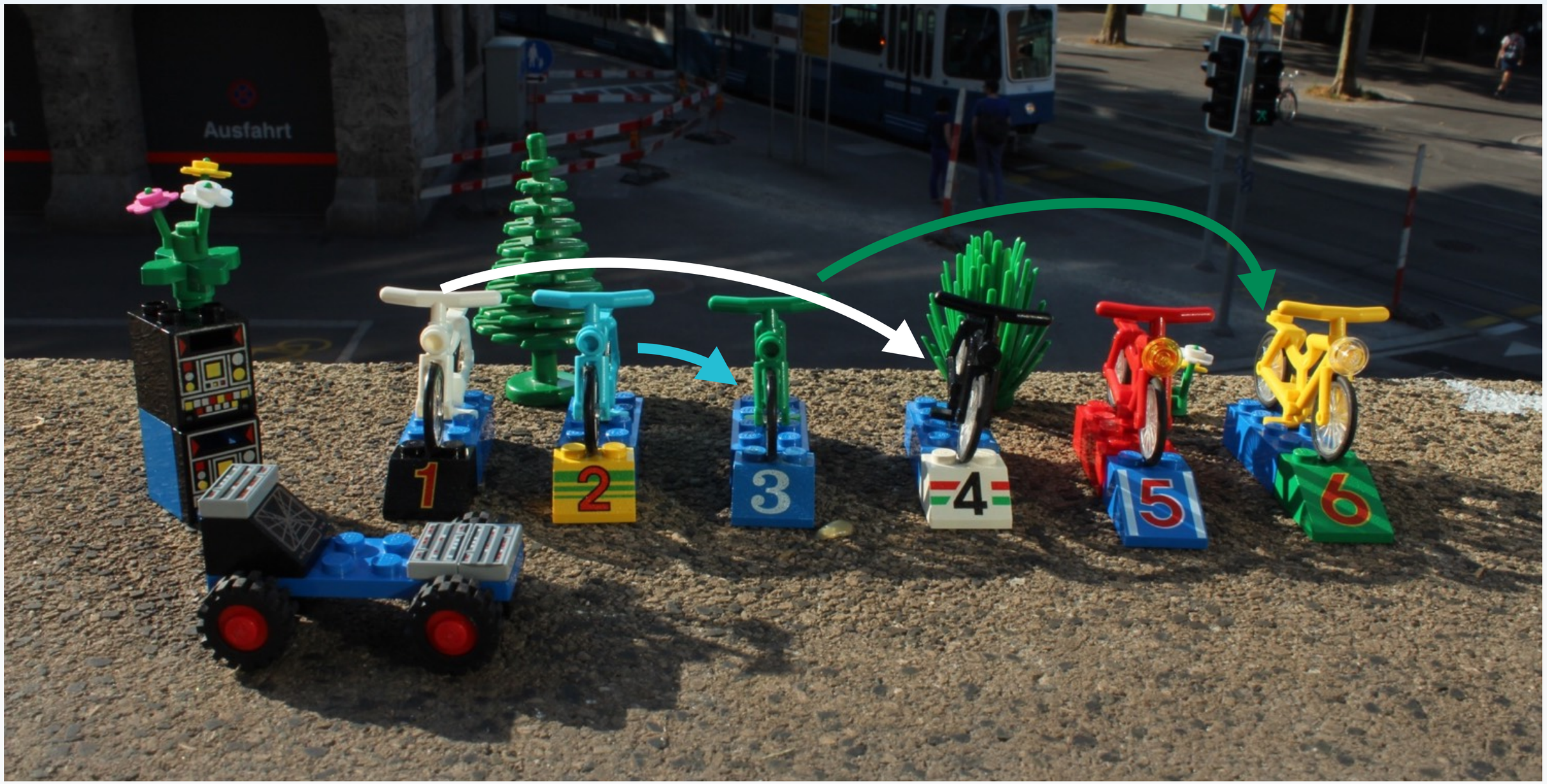

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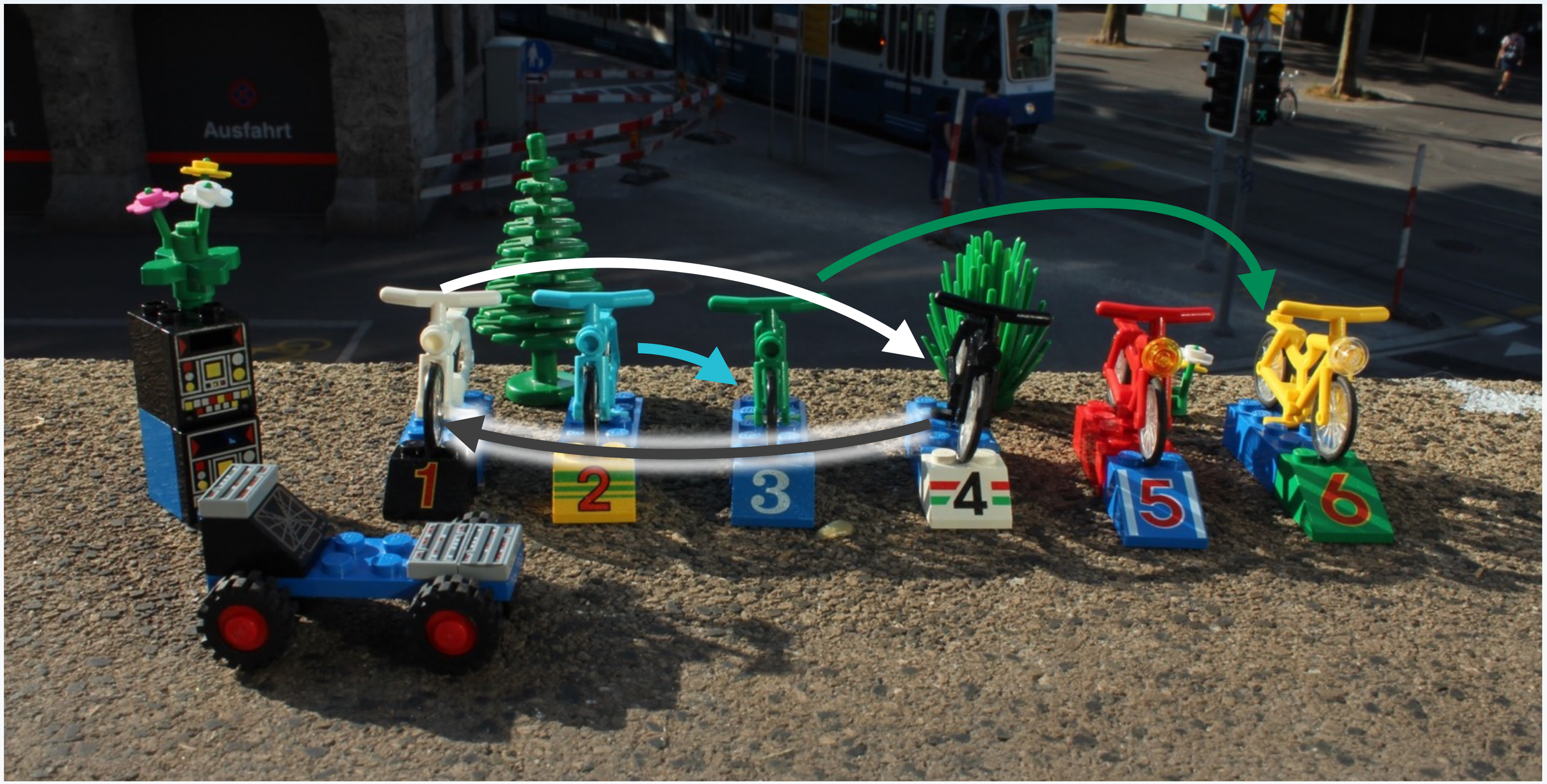

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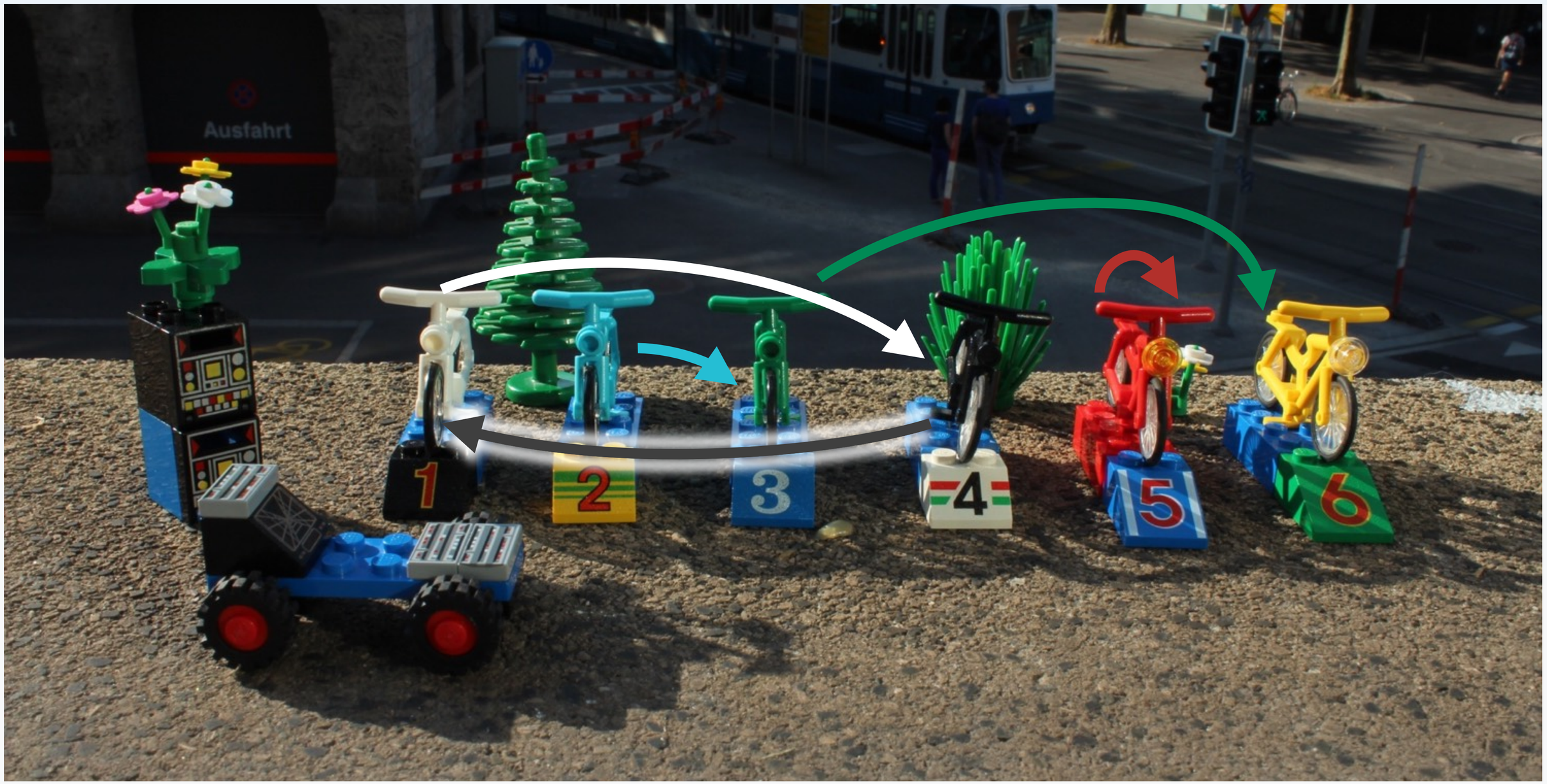

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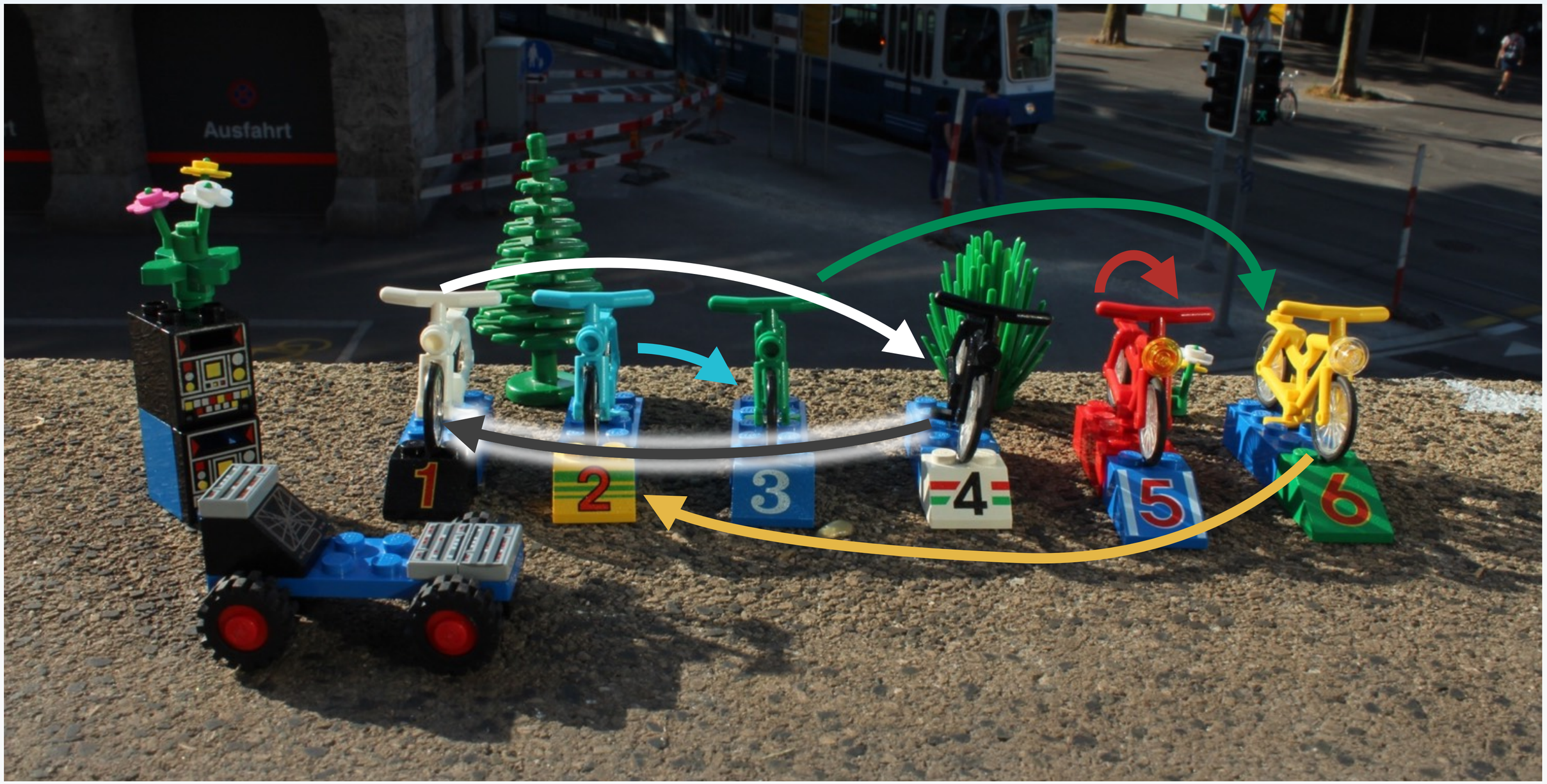

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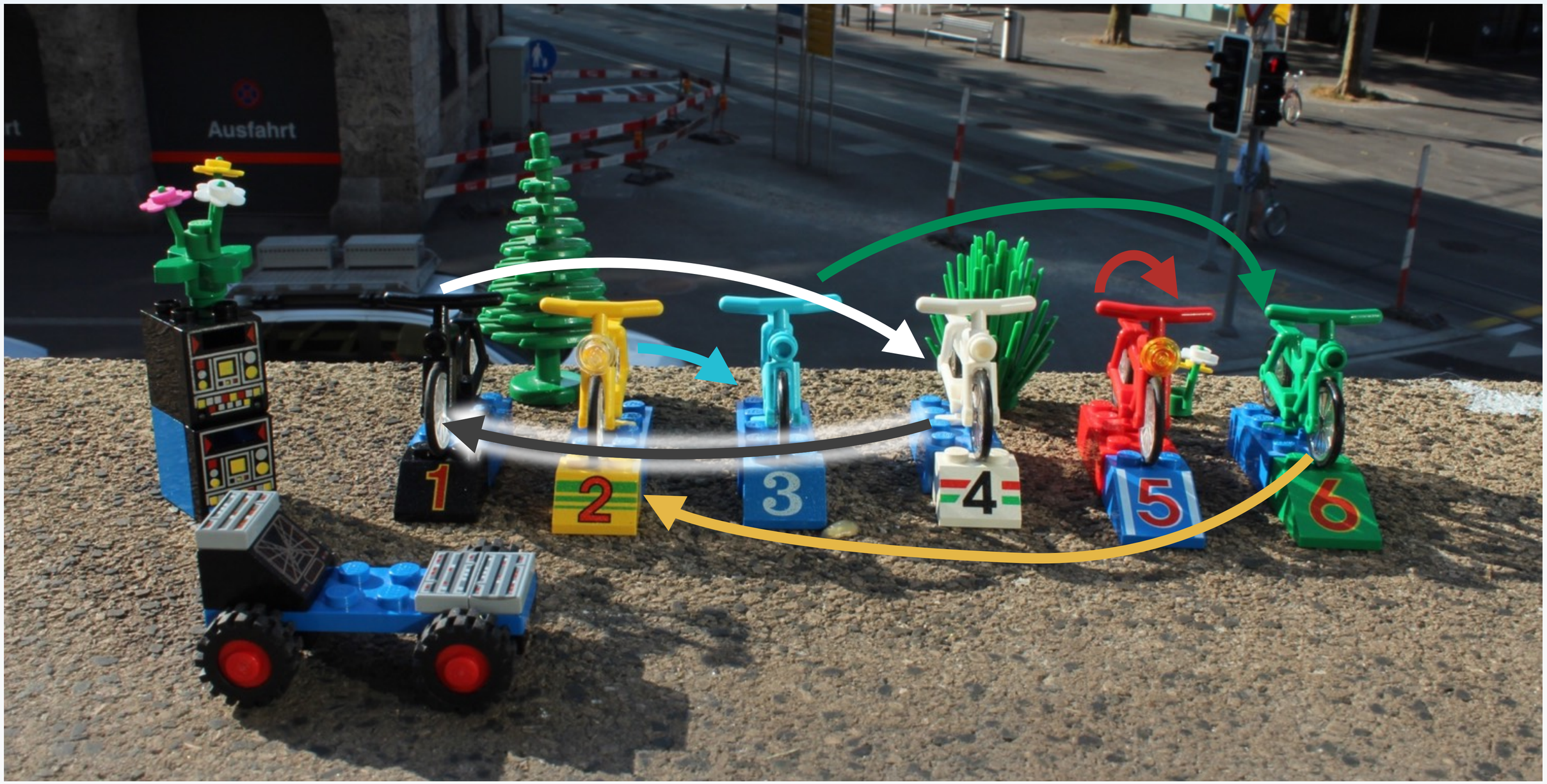

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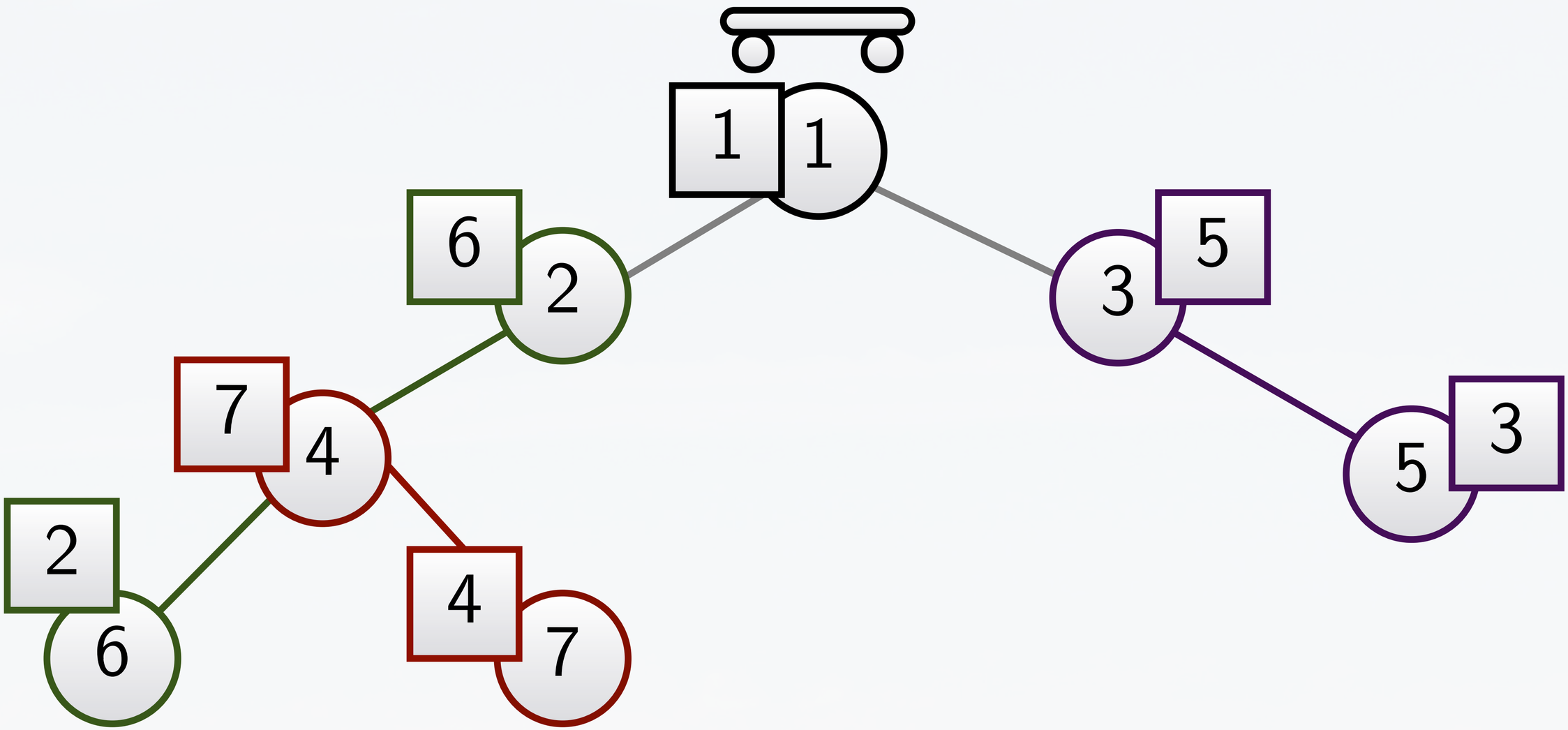
# Sorting Problem

# Example





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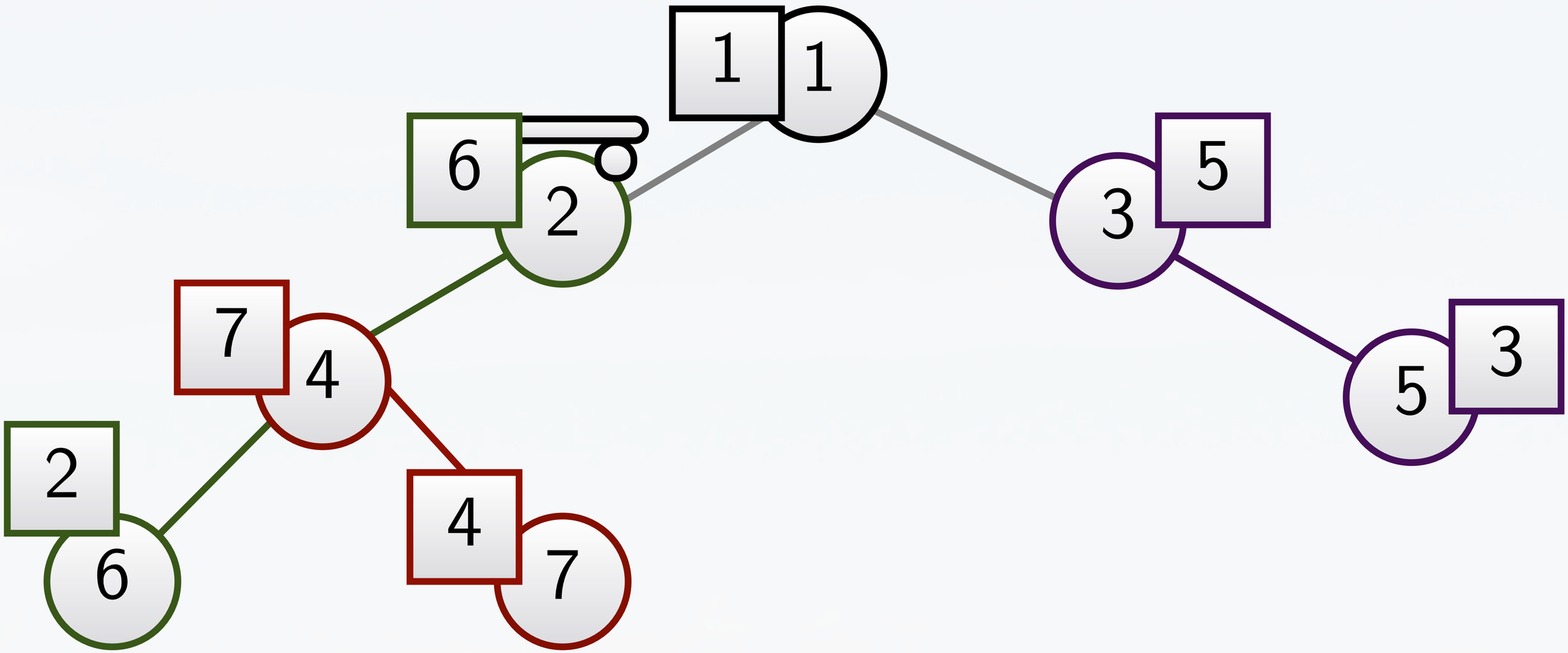

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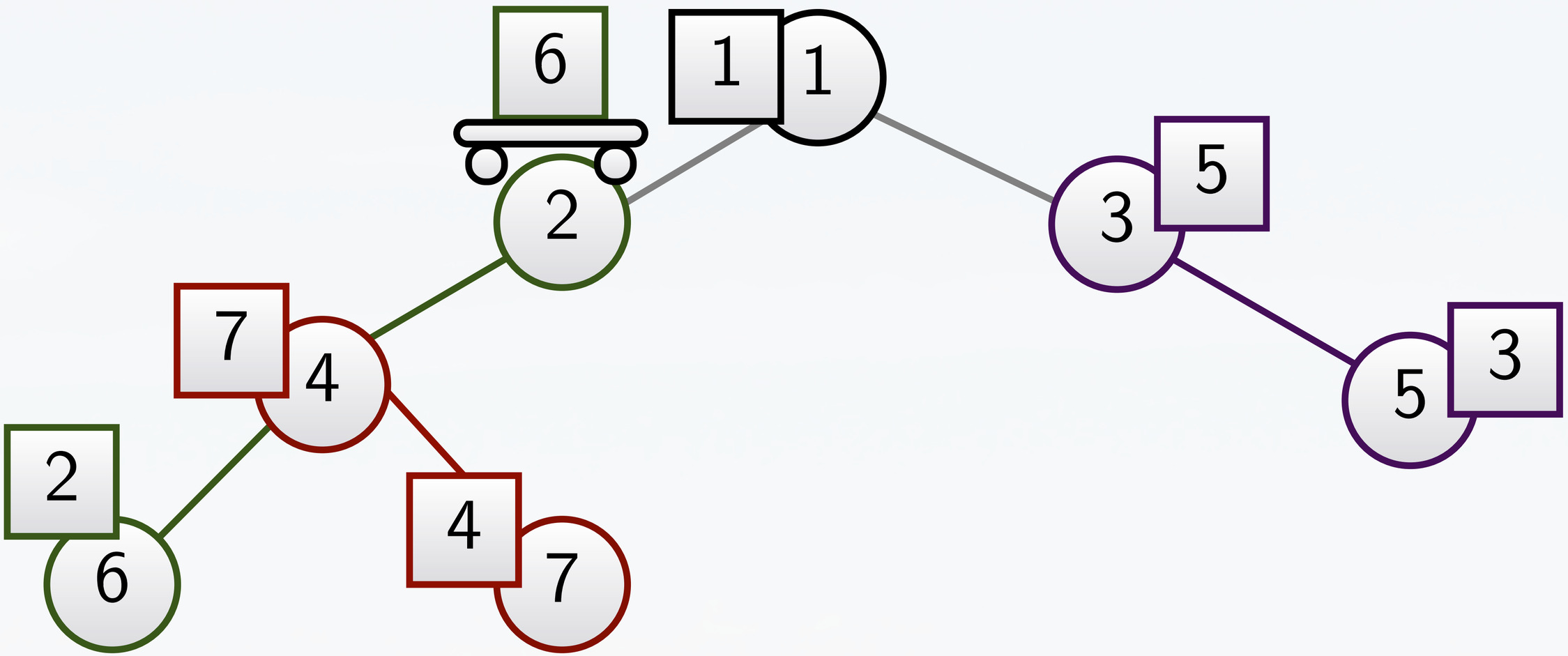

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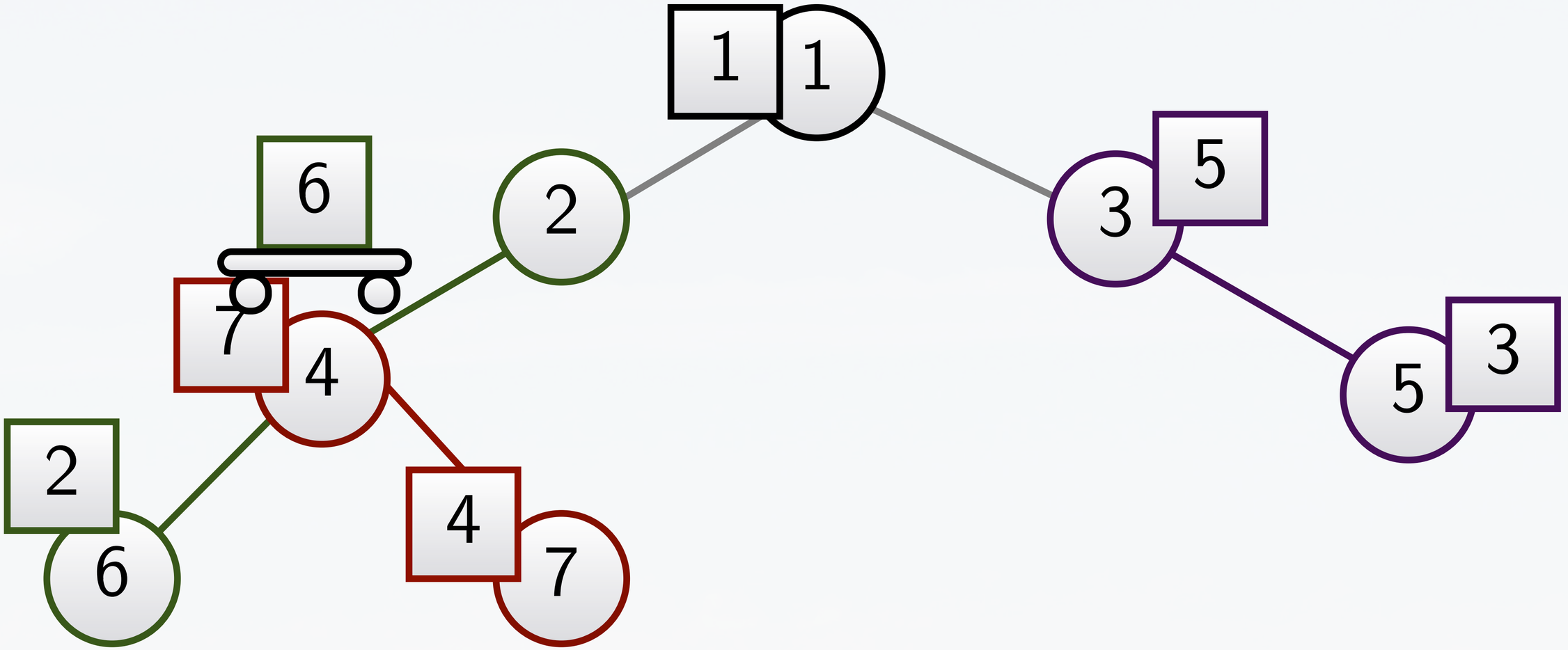

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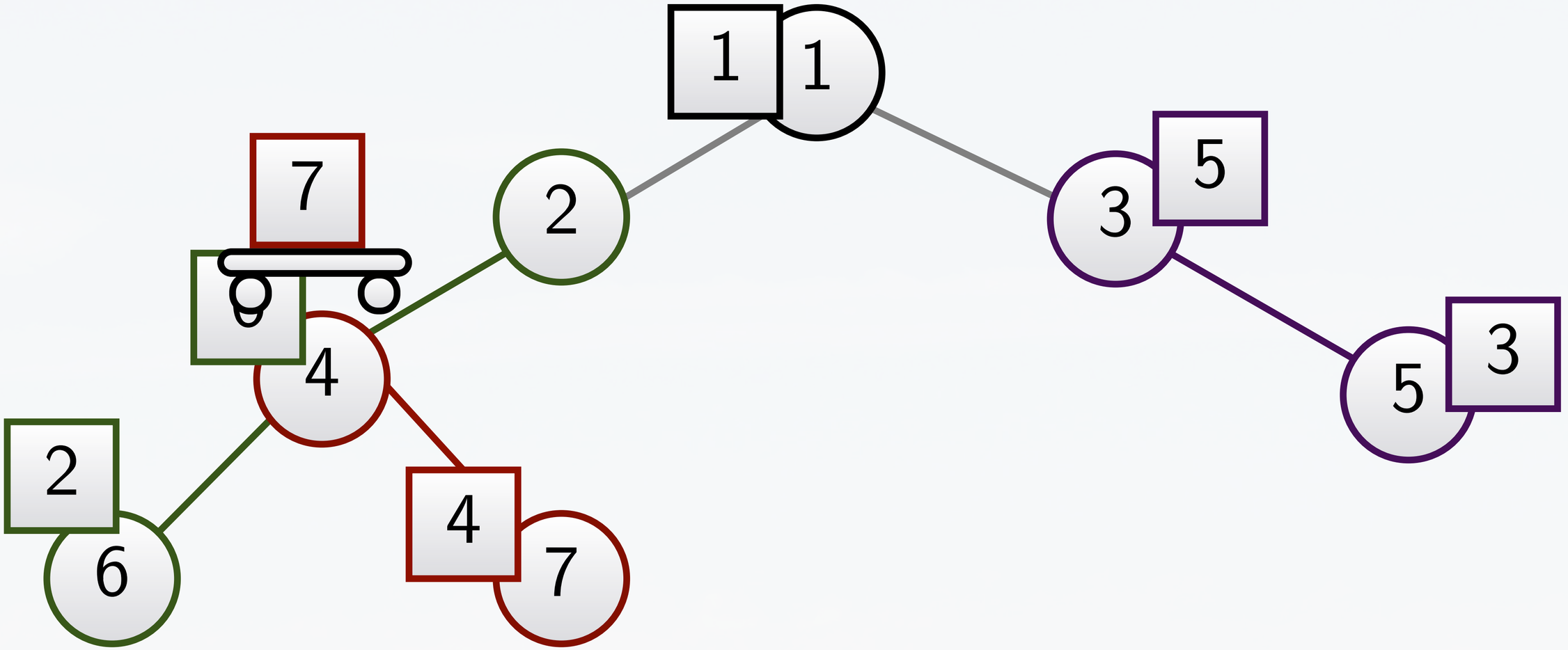

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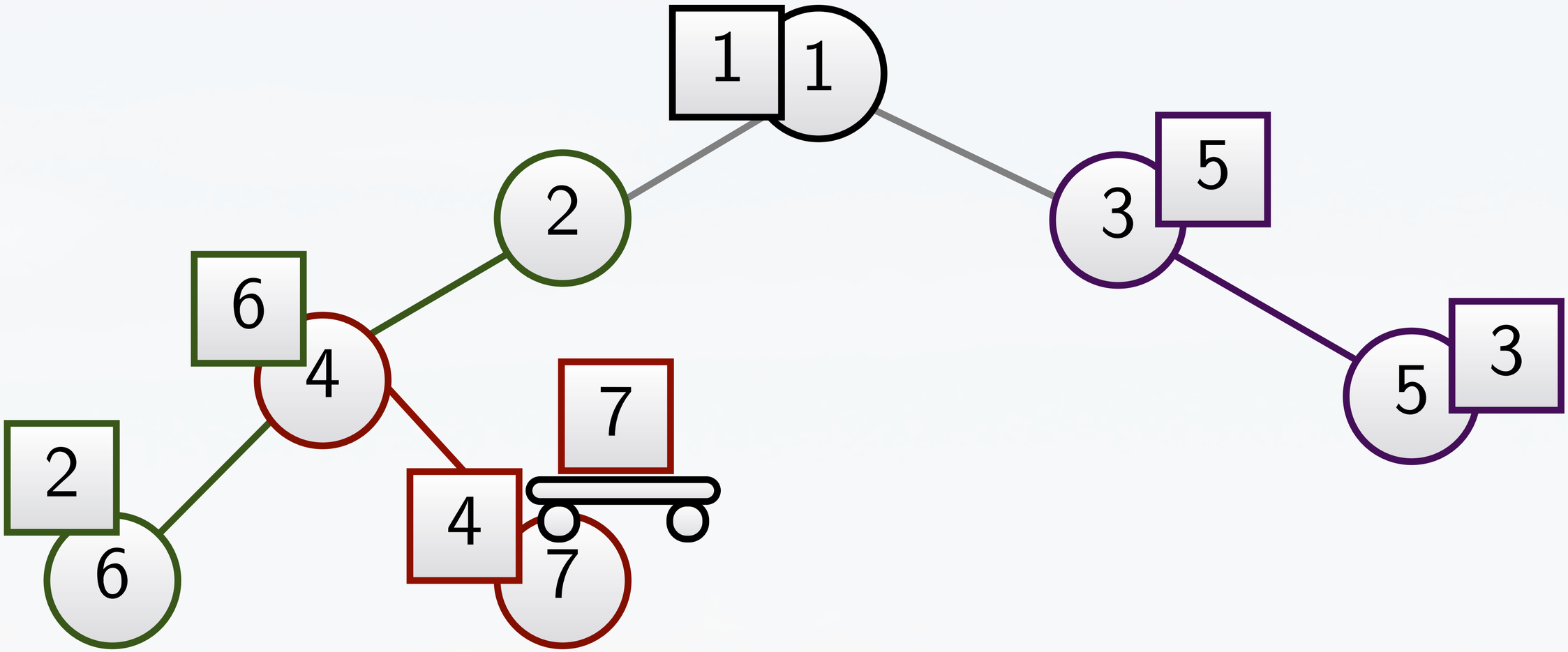

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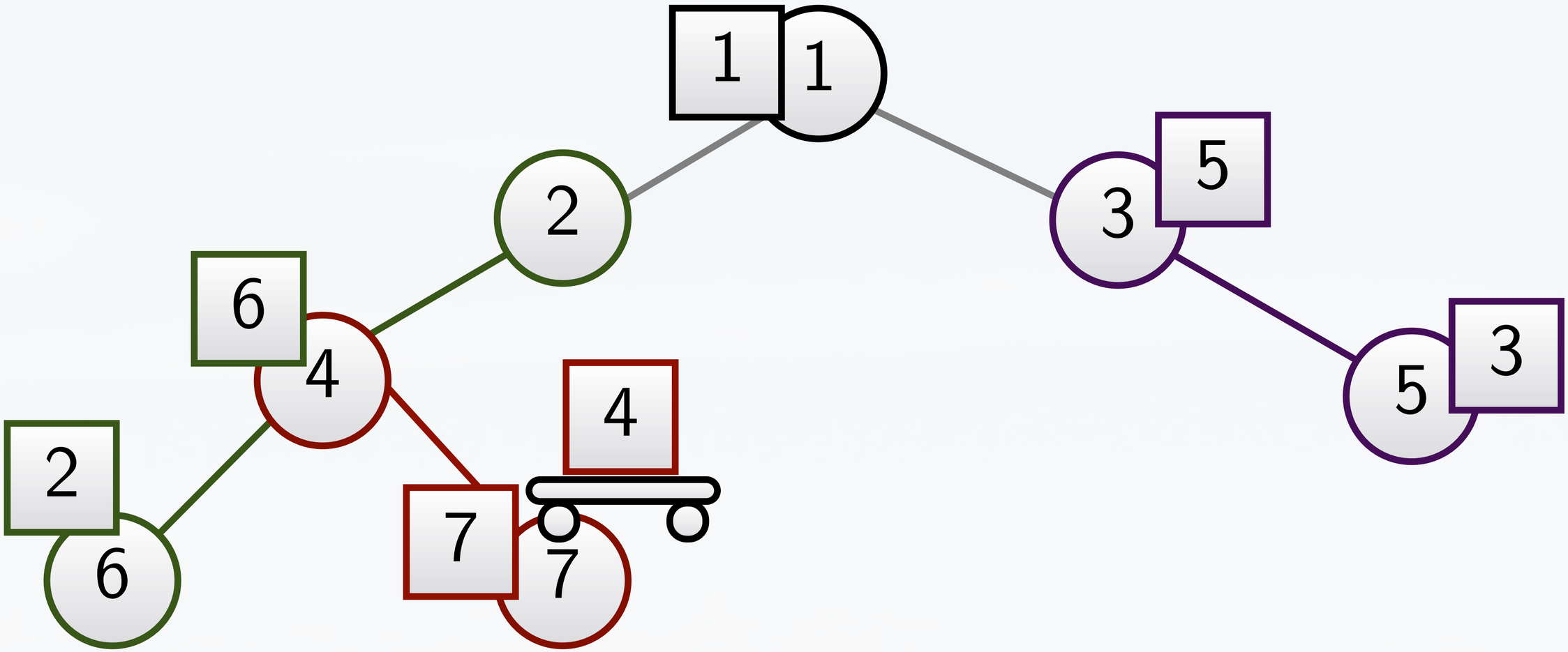

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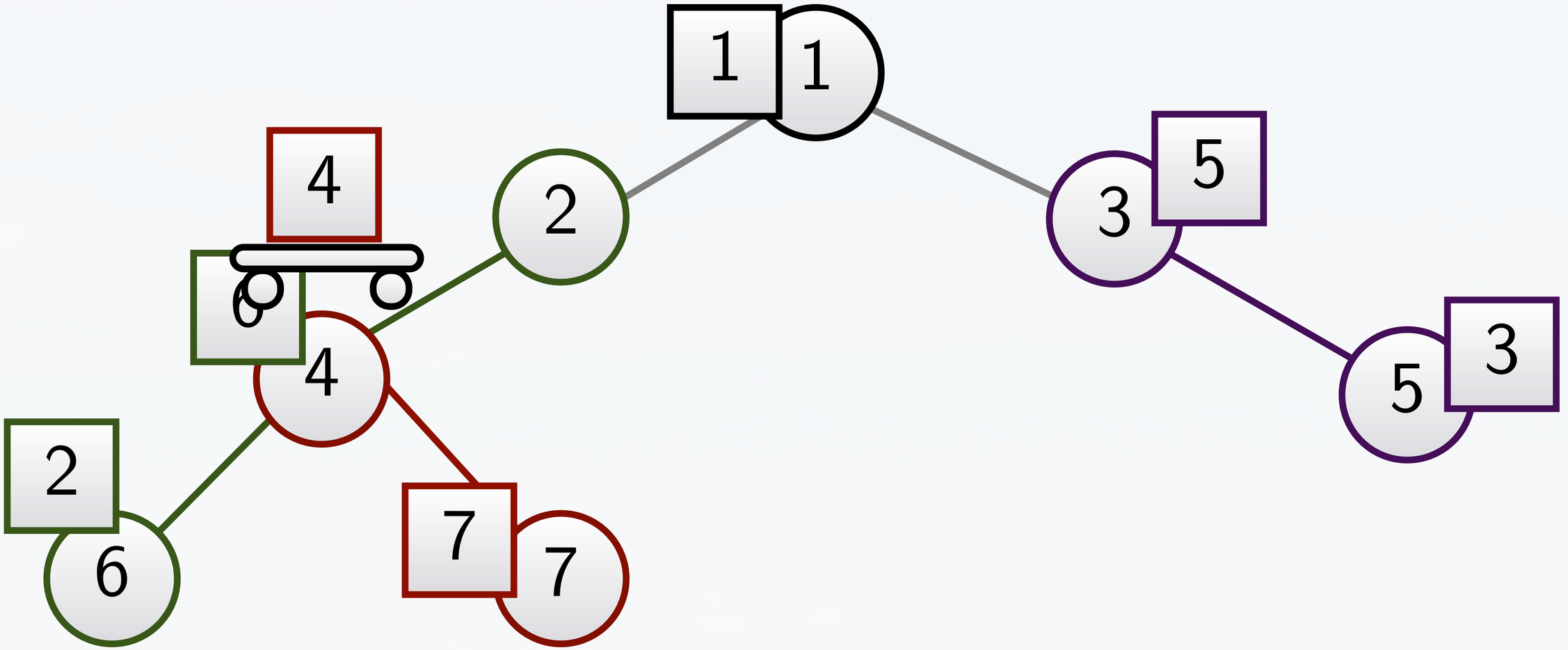

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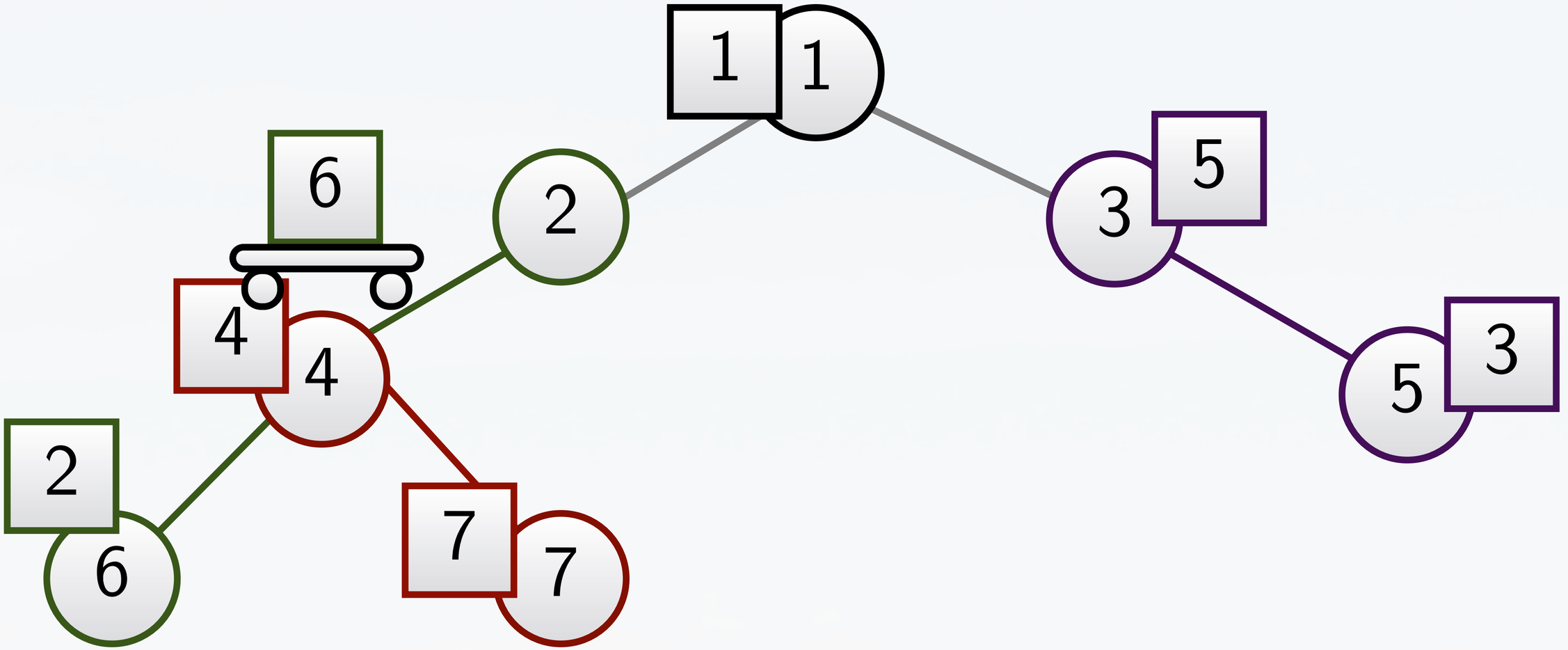

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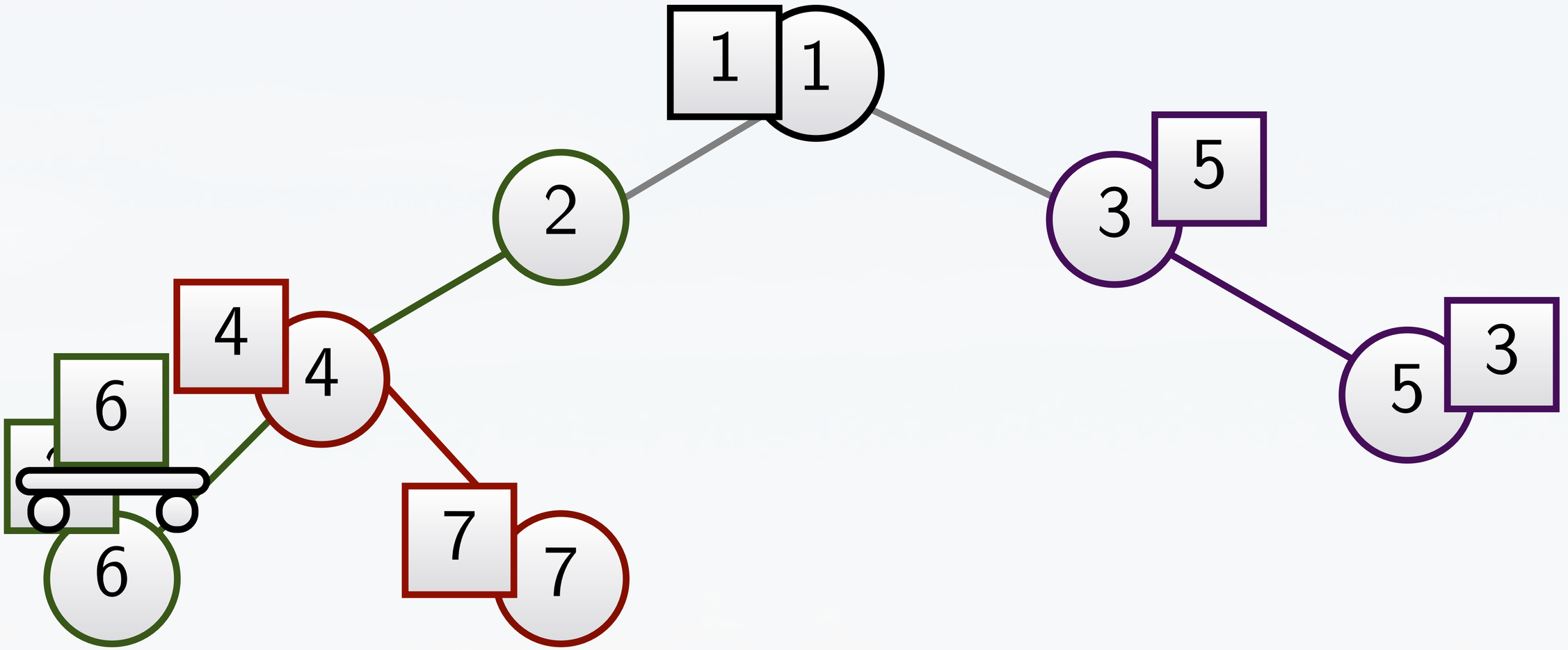

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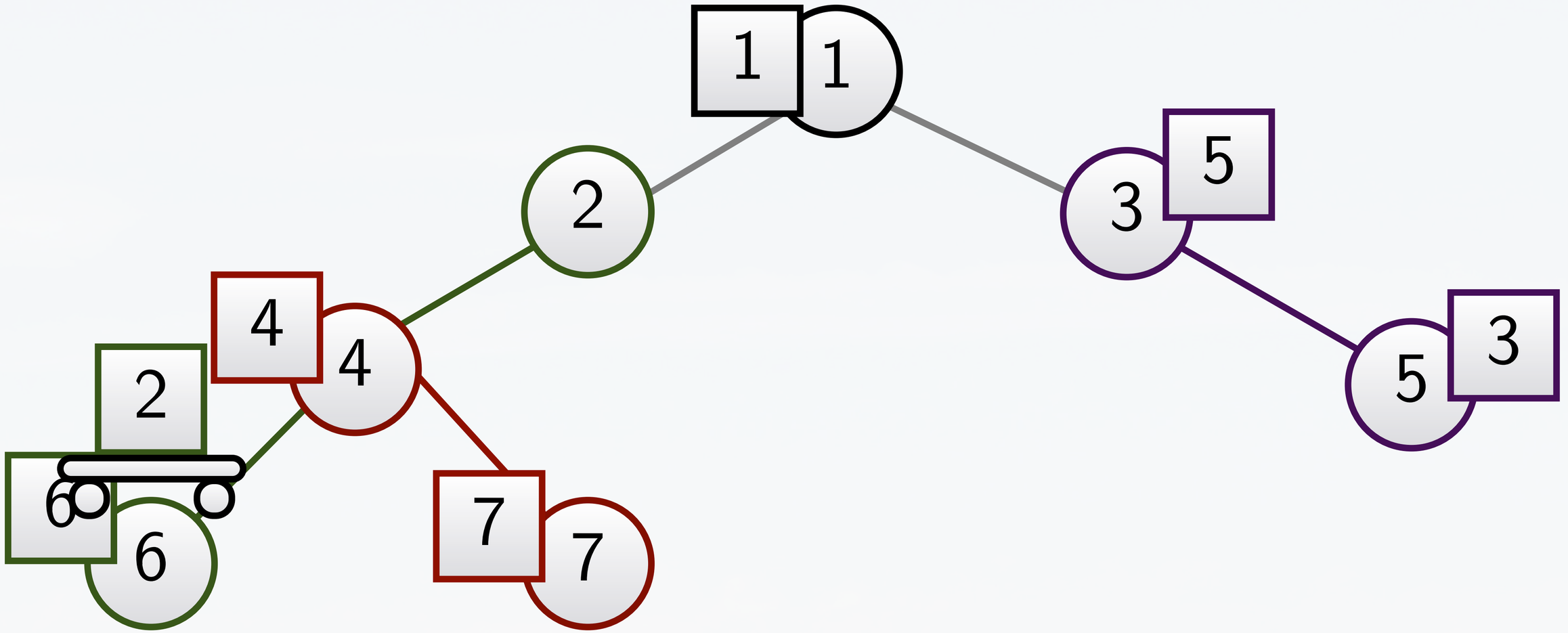

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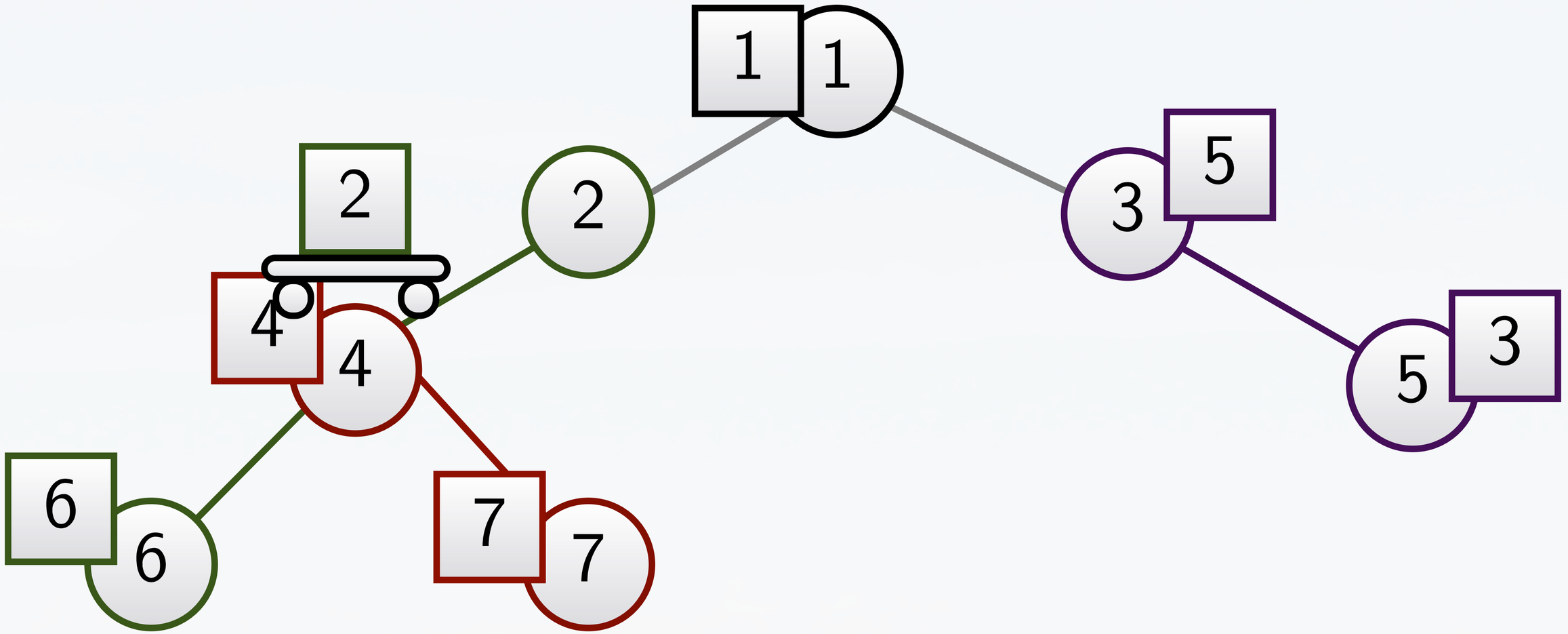

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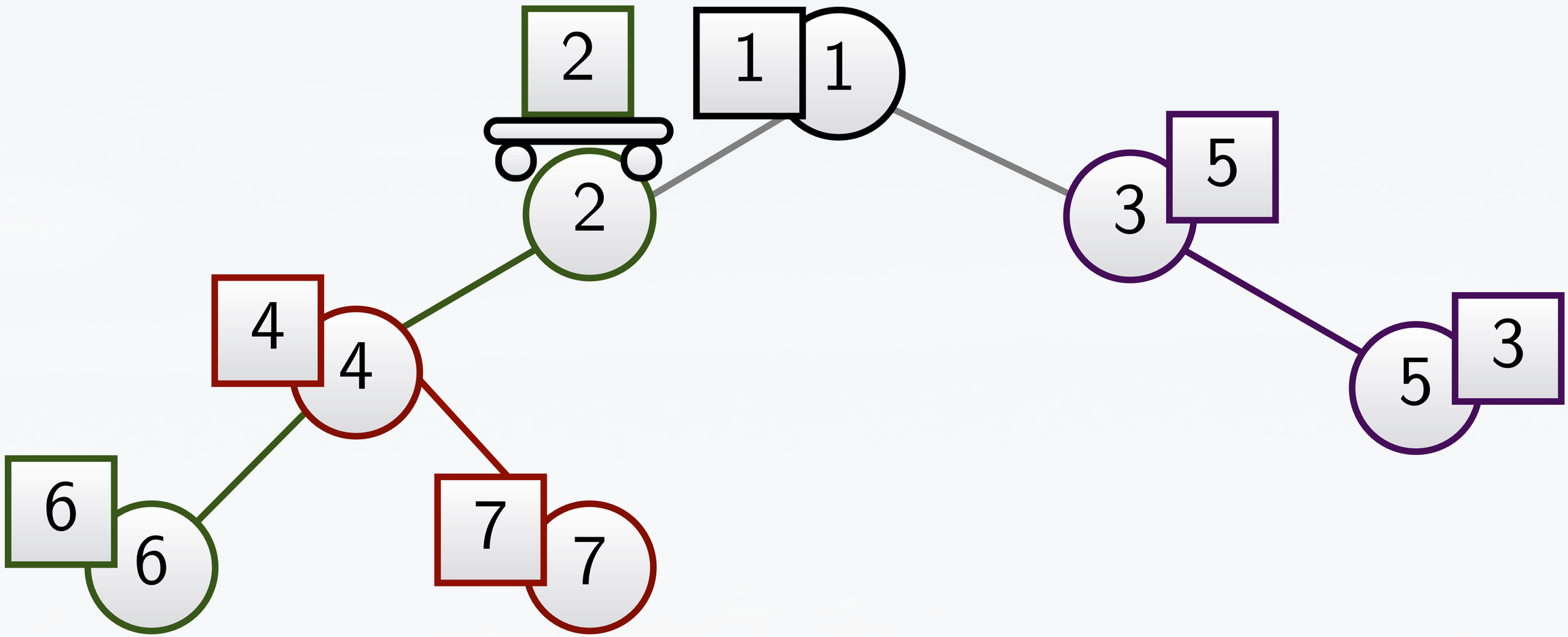

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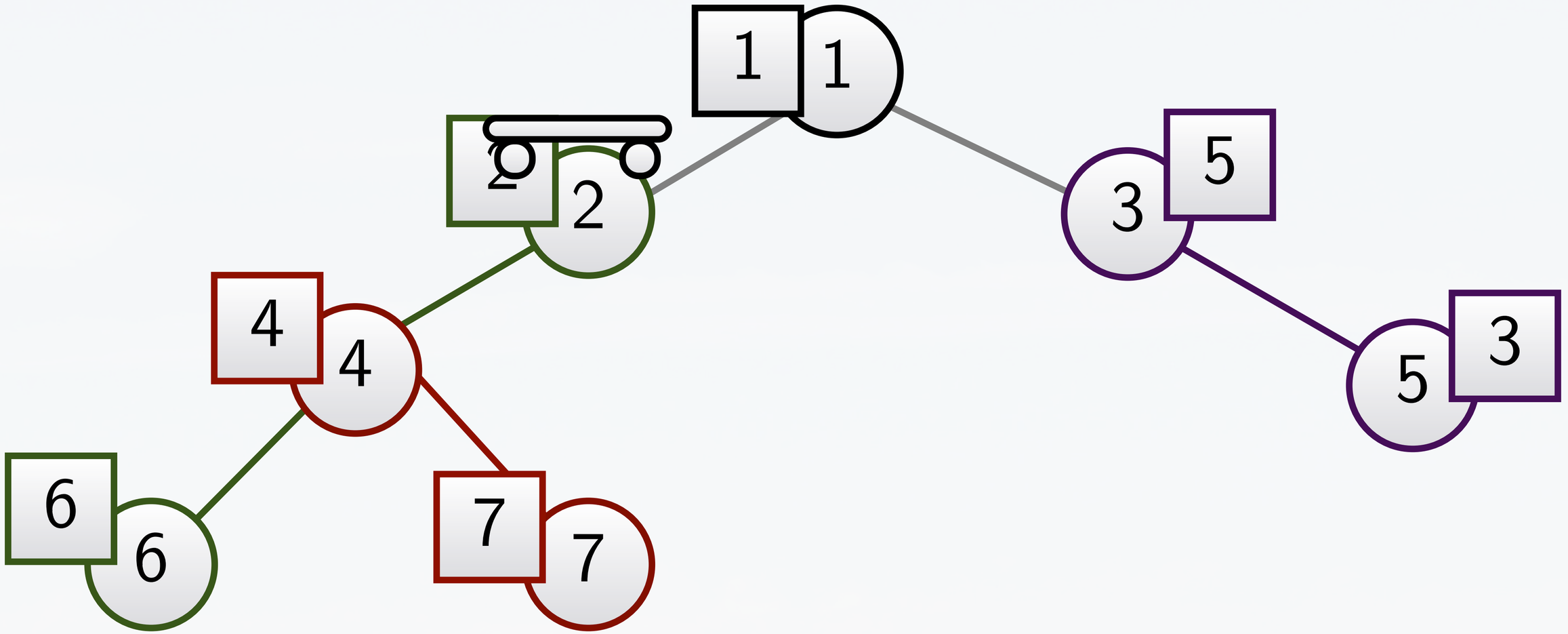

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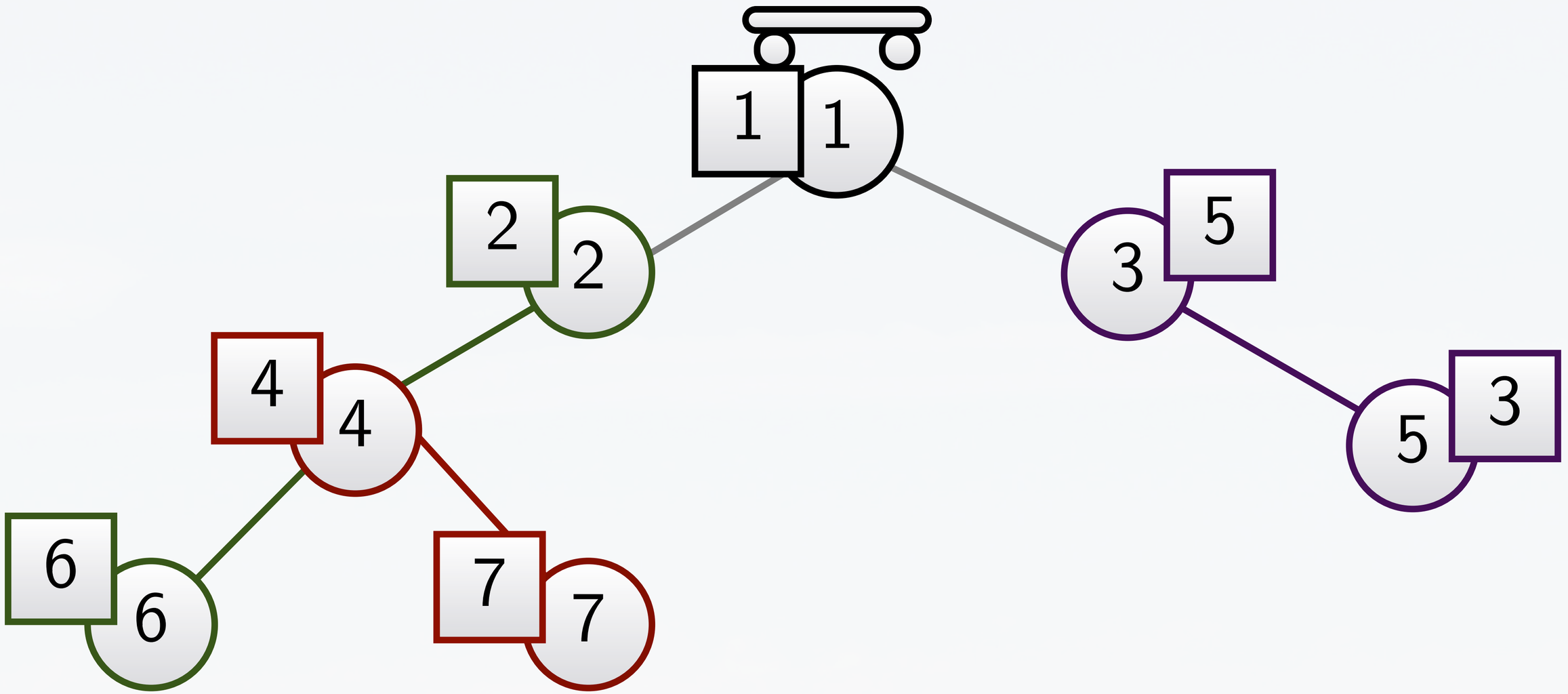

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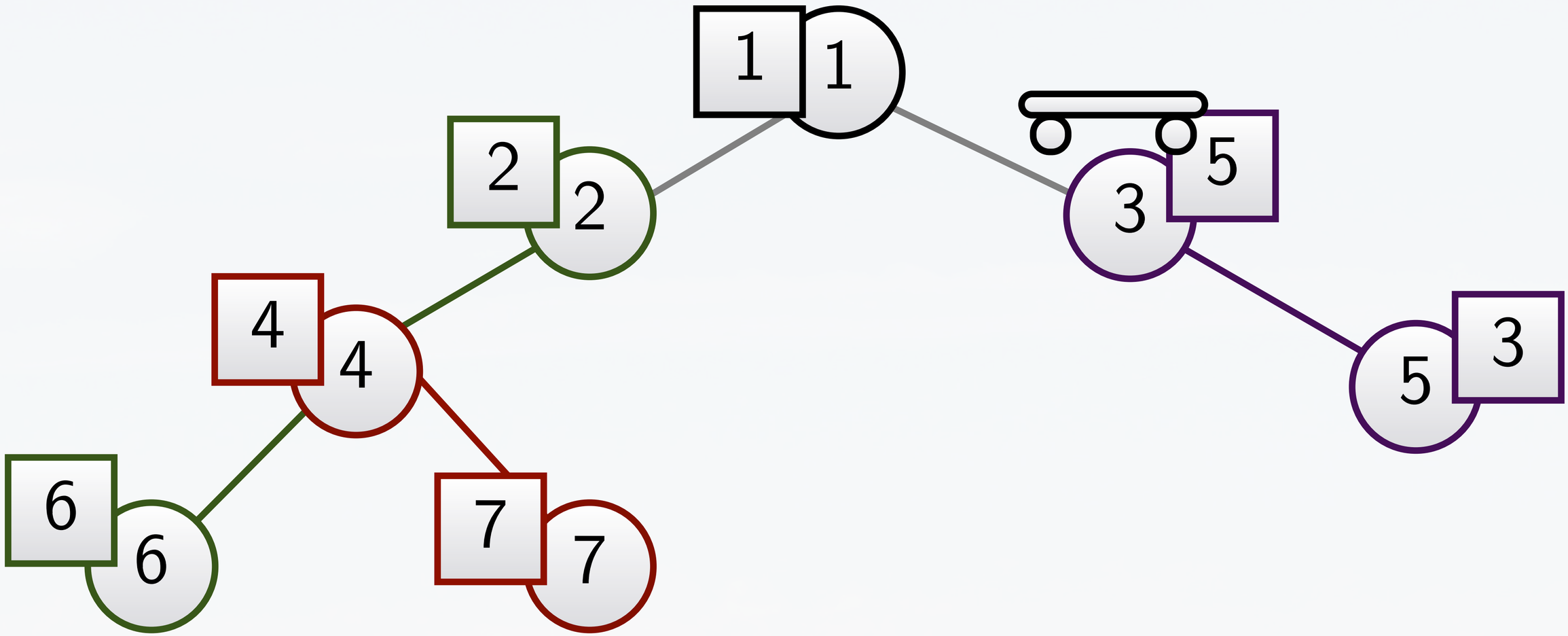

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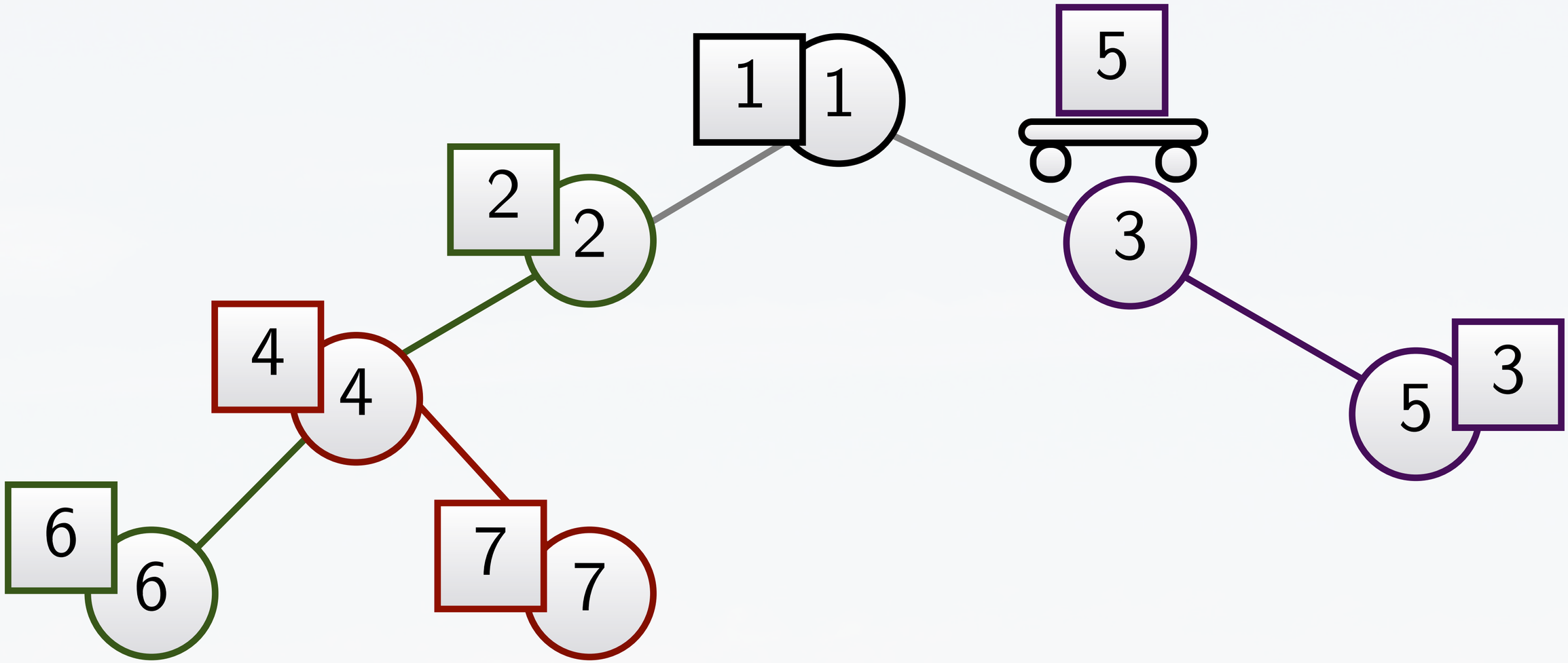

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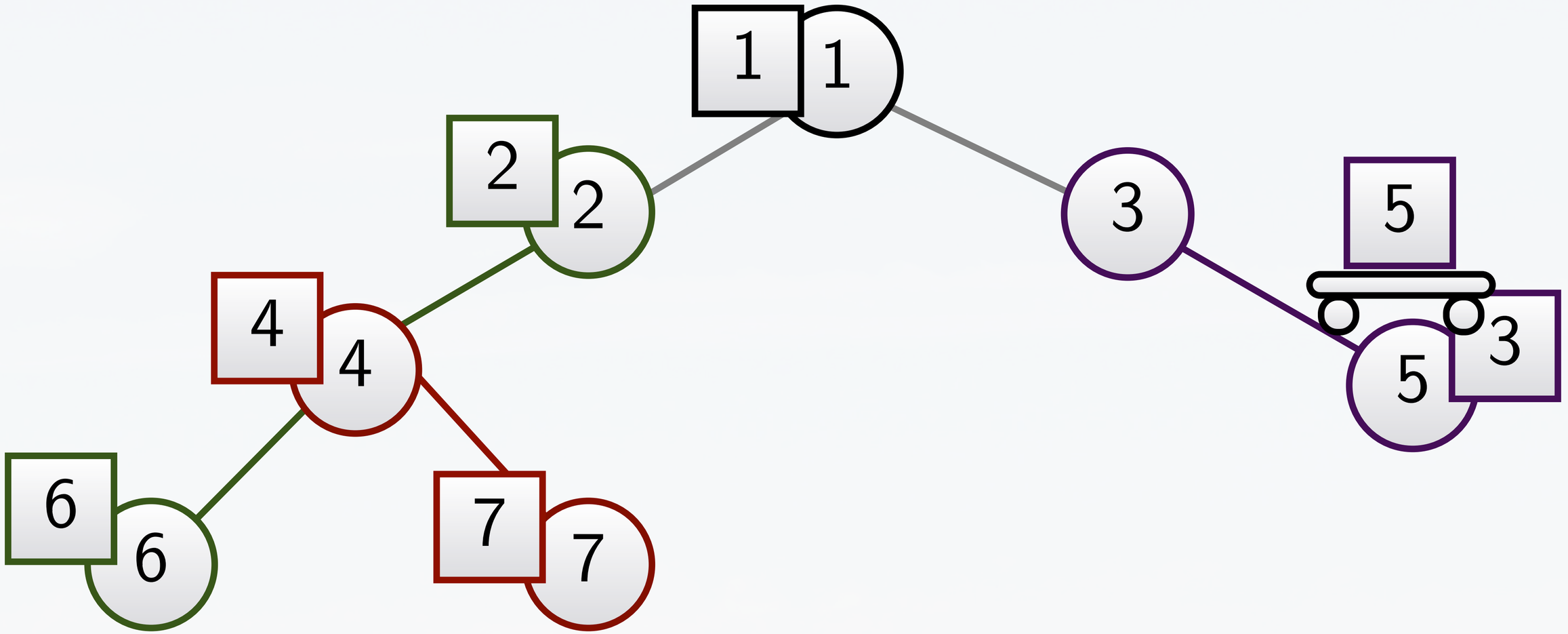

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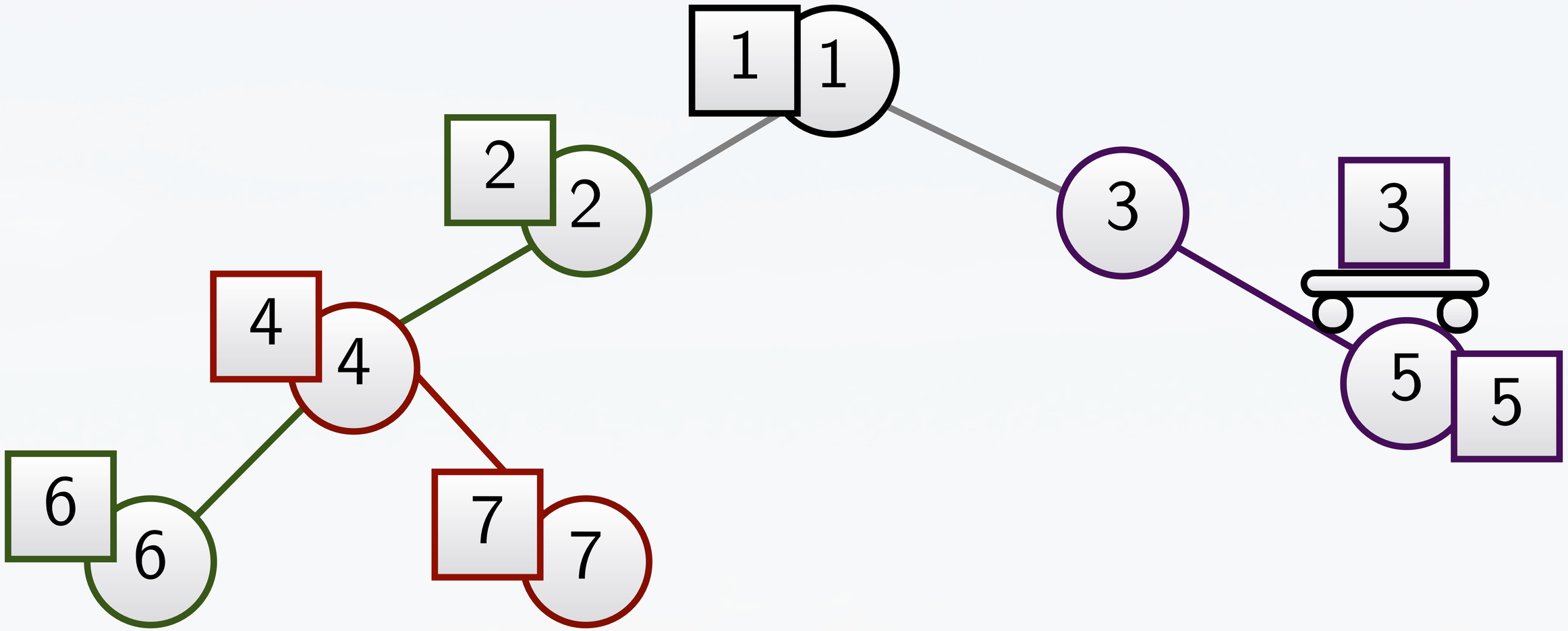

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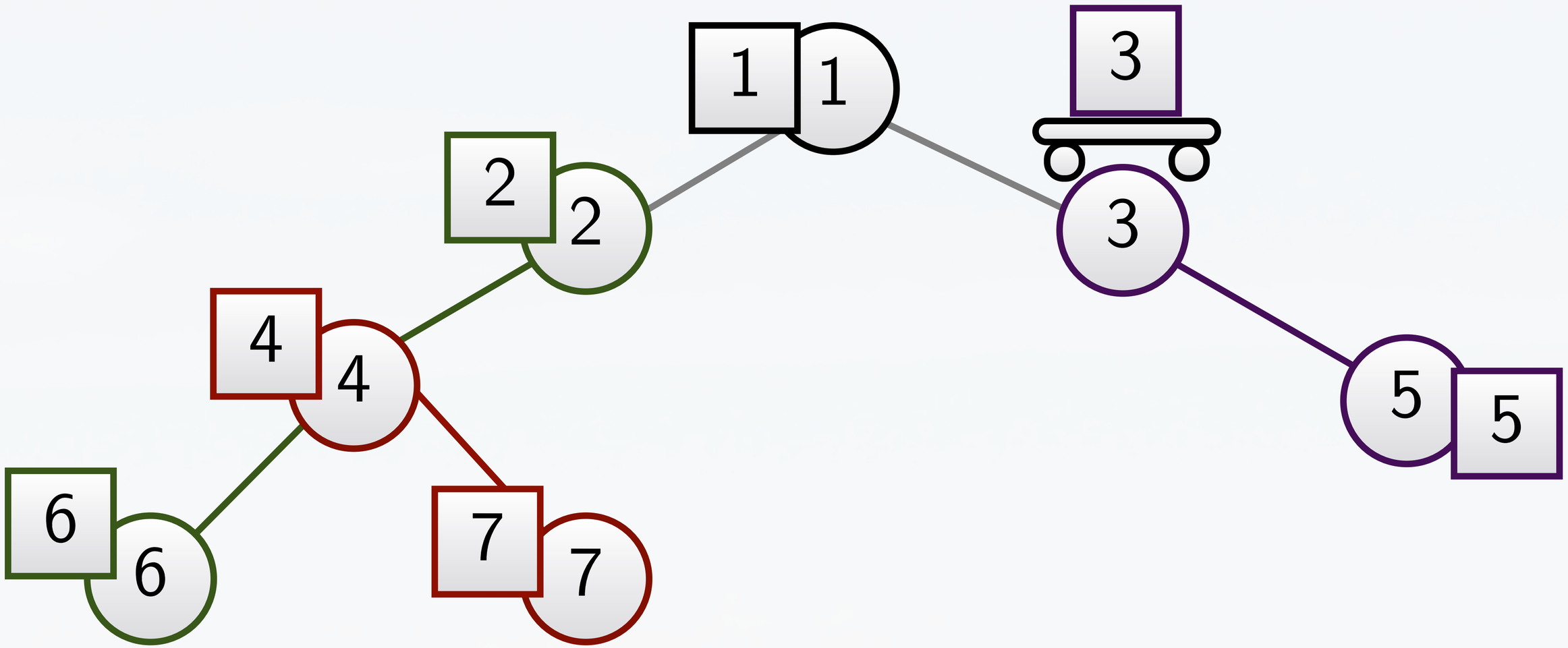

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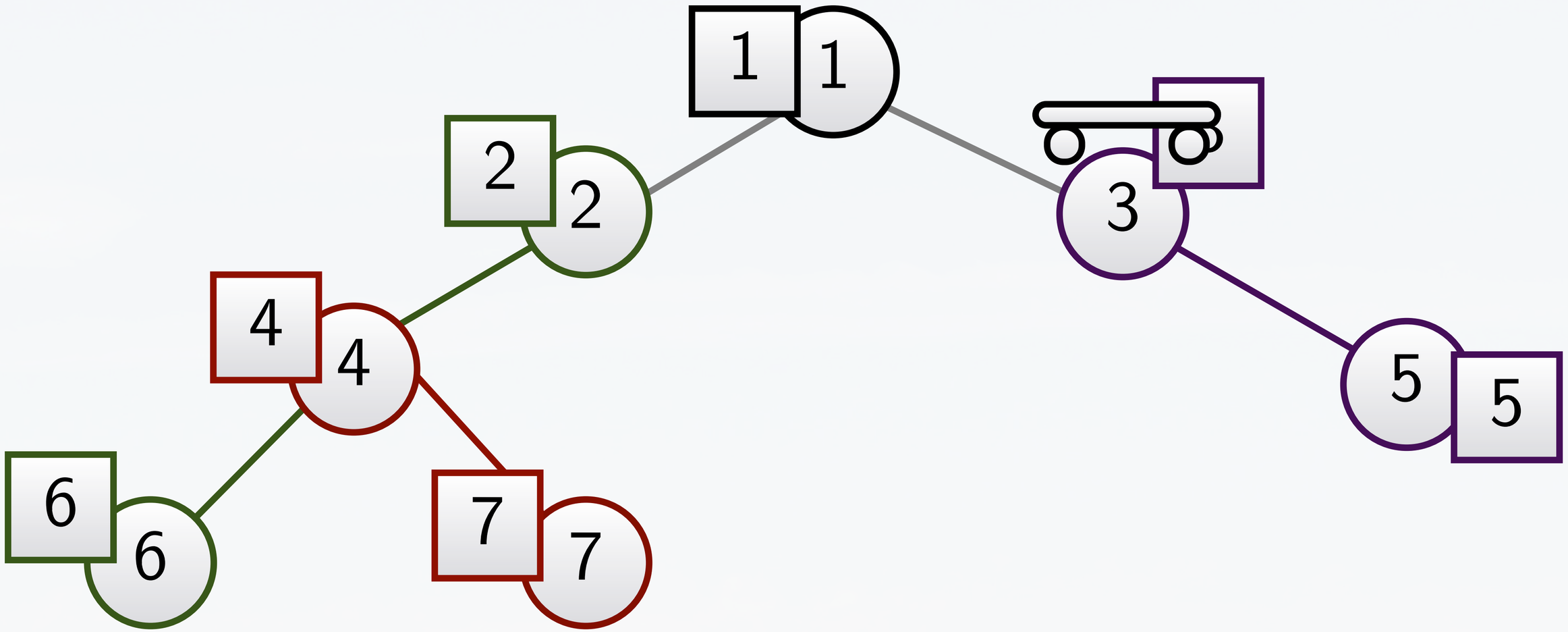

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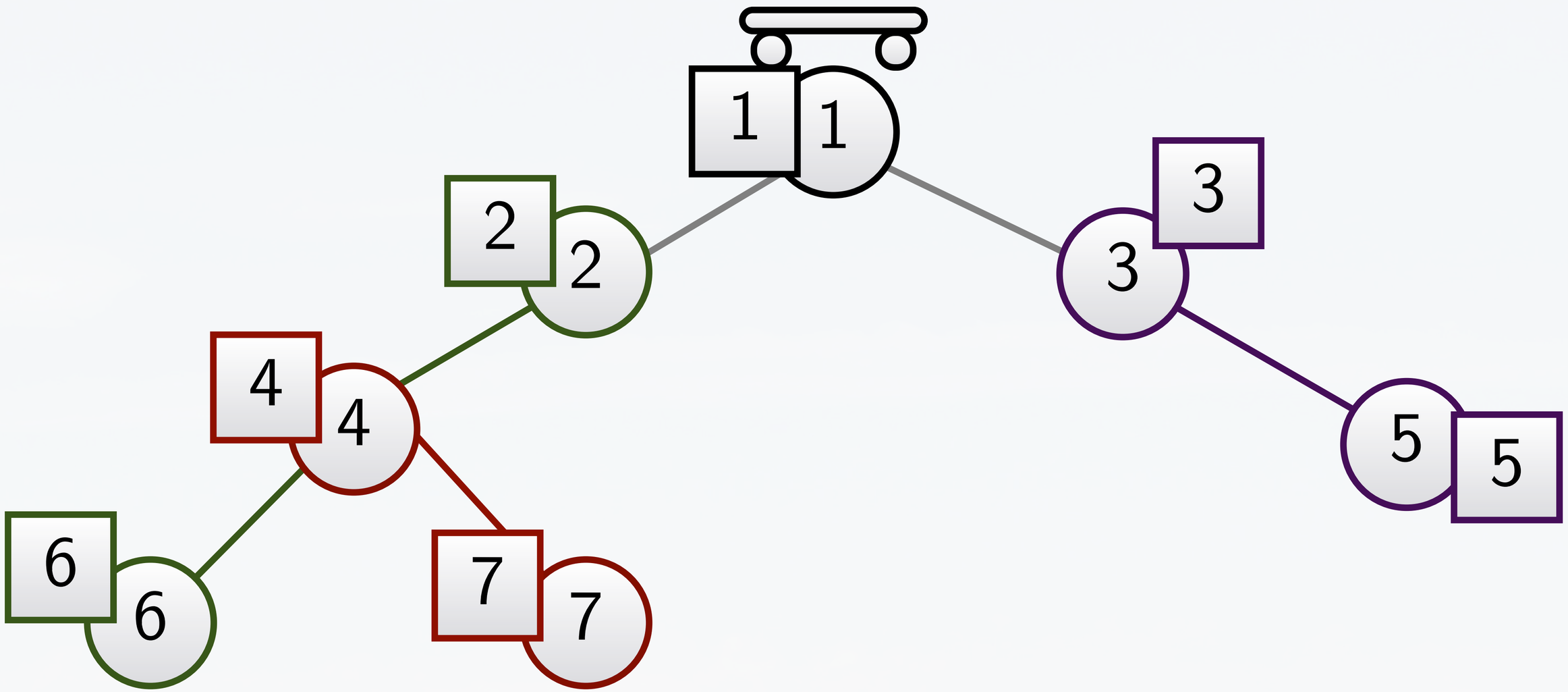

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*Definition:* ***essential step***

A step is essential, if it moves a box closer to its target position than it was in any previous state.

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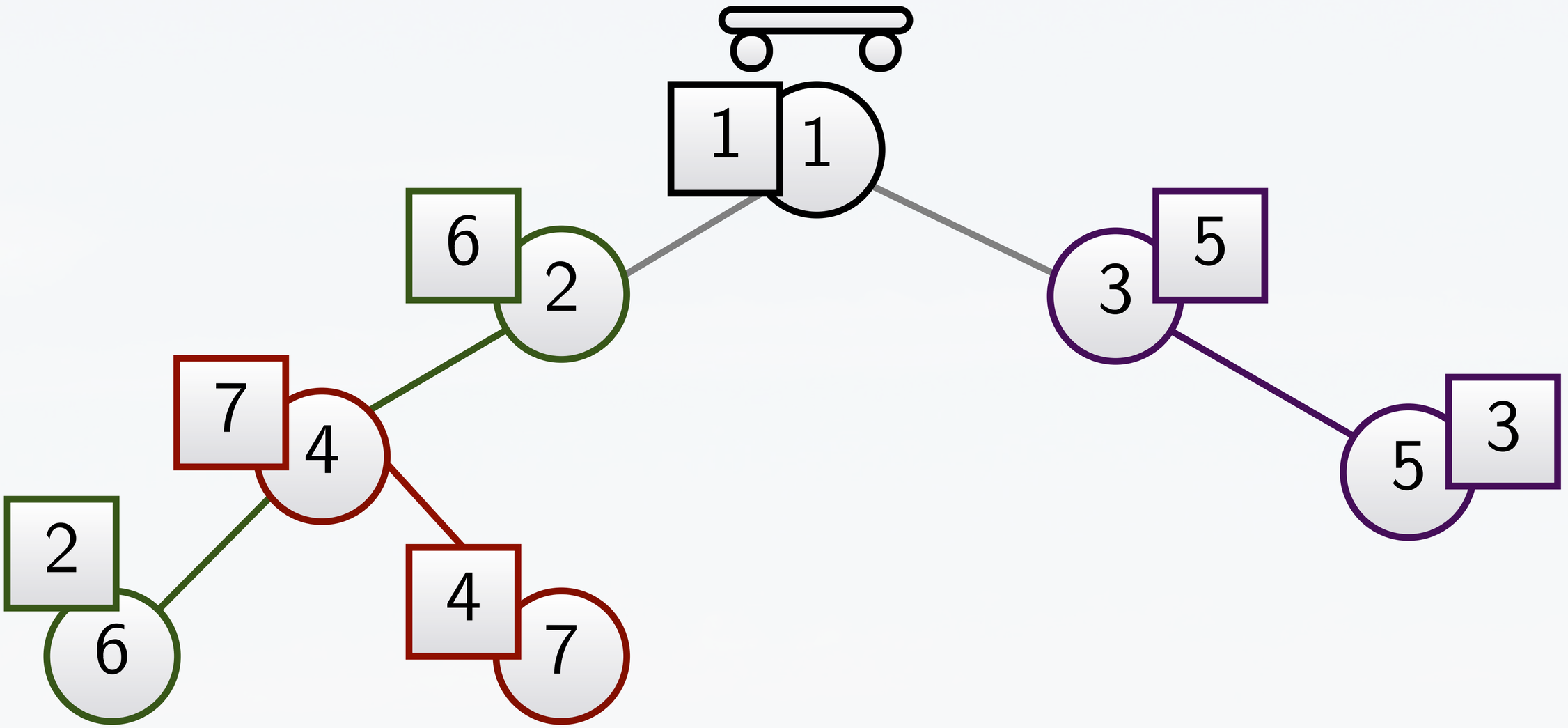
*Lemma: (Lower bound by essential steps)*

Every sorting walk contains at least $d(\pi) := \sum_{i=1}^{n} d(i, \pi(i))$ steps.

# General Bounds



$$d(\pi) = 1 + 1 + 1 + 1 + 2 + 2 = 8$$



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We can always find a shorting walk of length $d(\pi)+(2n-2)$.

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*Remaining:* How can we interleave the cycles to close this linear gap between the bounds?

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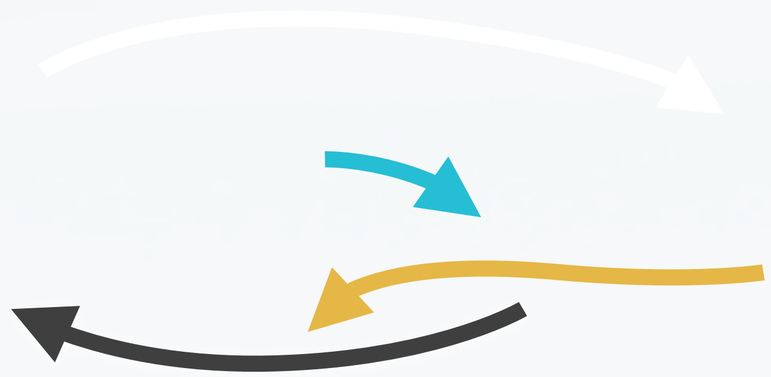
# On Paths



Path graph with the robot starting at the border.

Straightforward if they form a single cycle.



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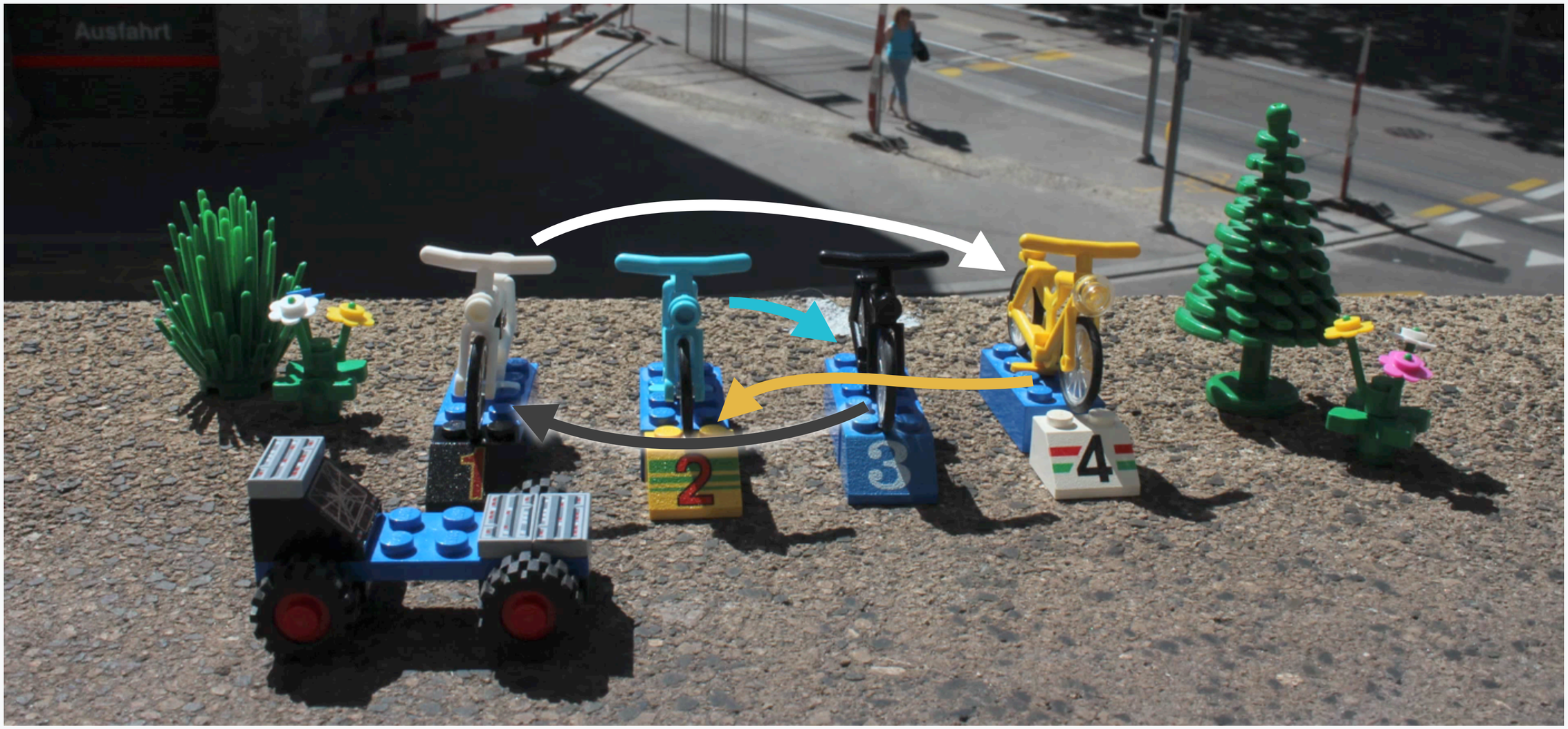
# On Paths



Path graph with the robot starting at the border.

Straightforward if they form a single cycle.



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# On Paths



*Observation:* Overlapping cycles can be sorted without non-essential steps.

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# On Paths



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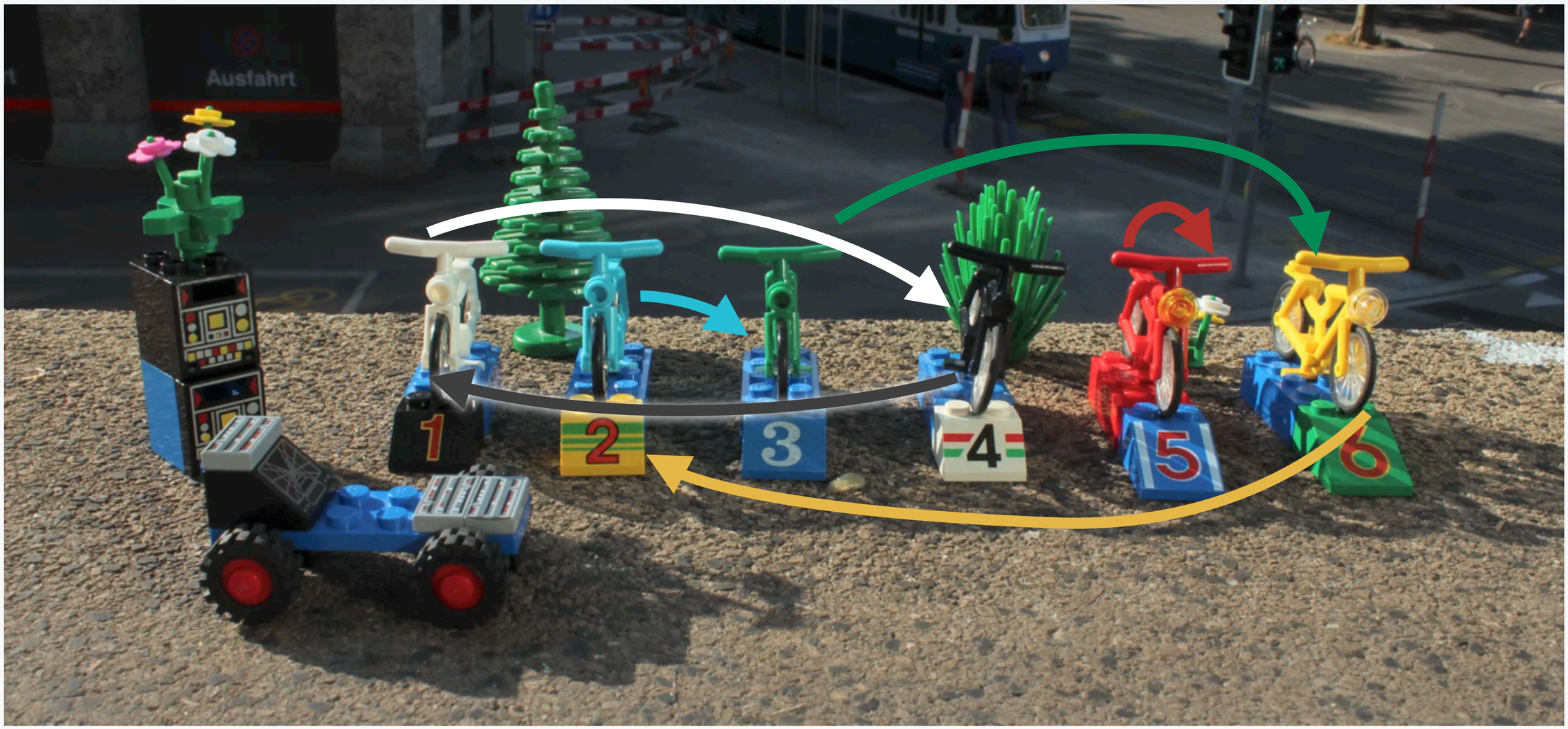
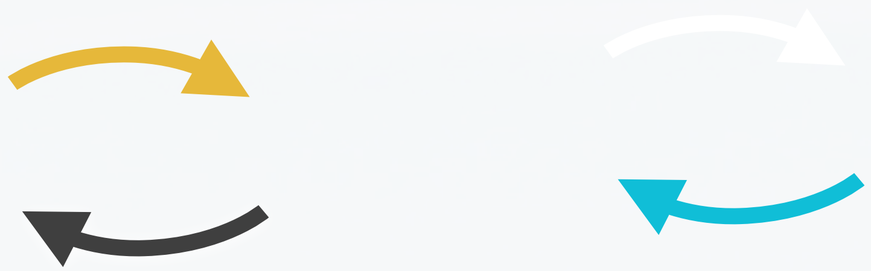


# On Paths



*Observation:* Non-essential steps are only needed across edges that are between cycles of $\pi$.



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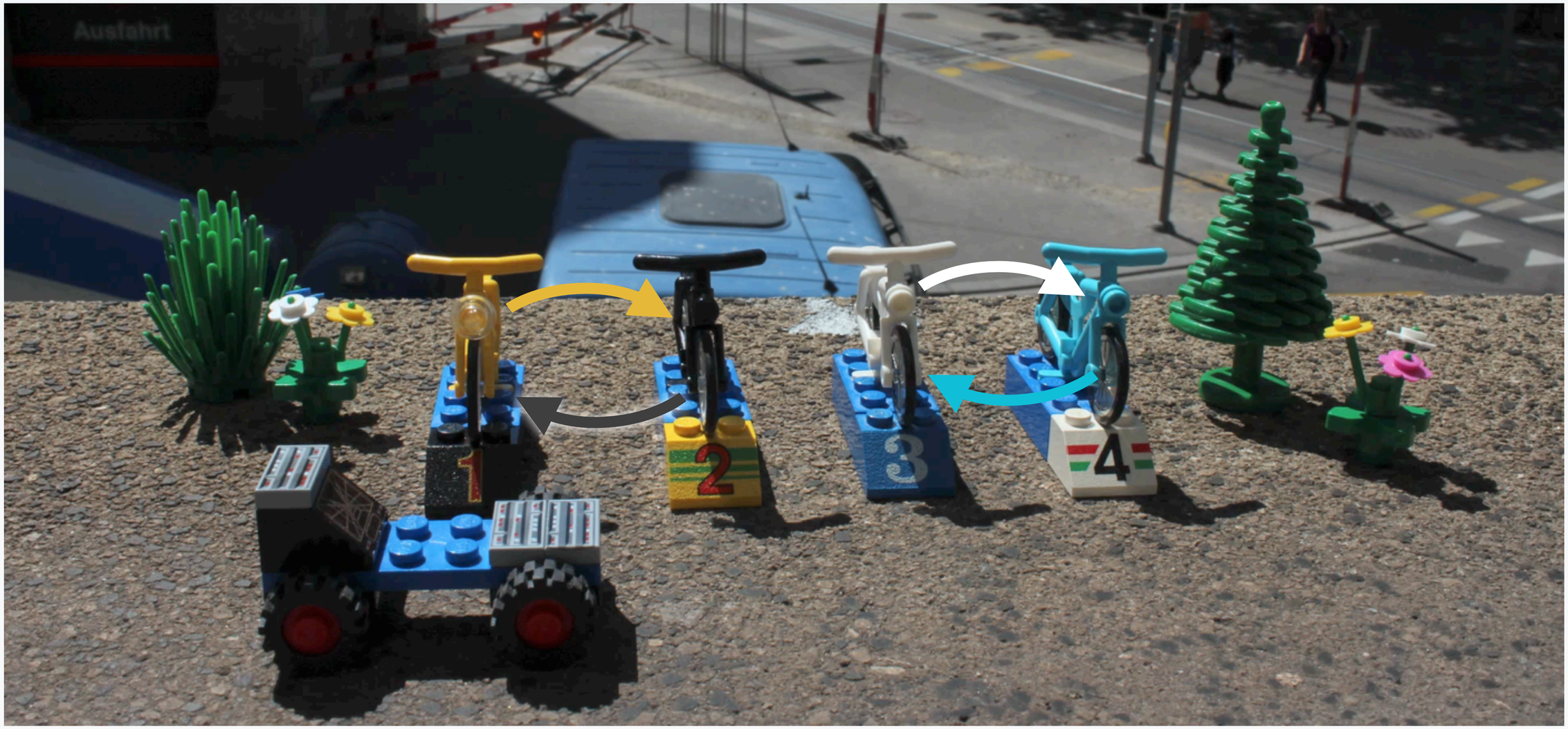

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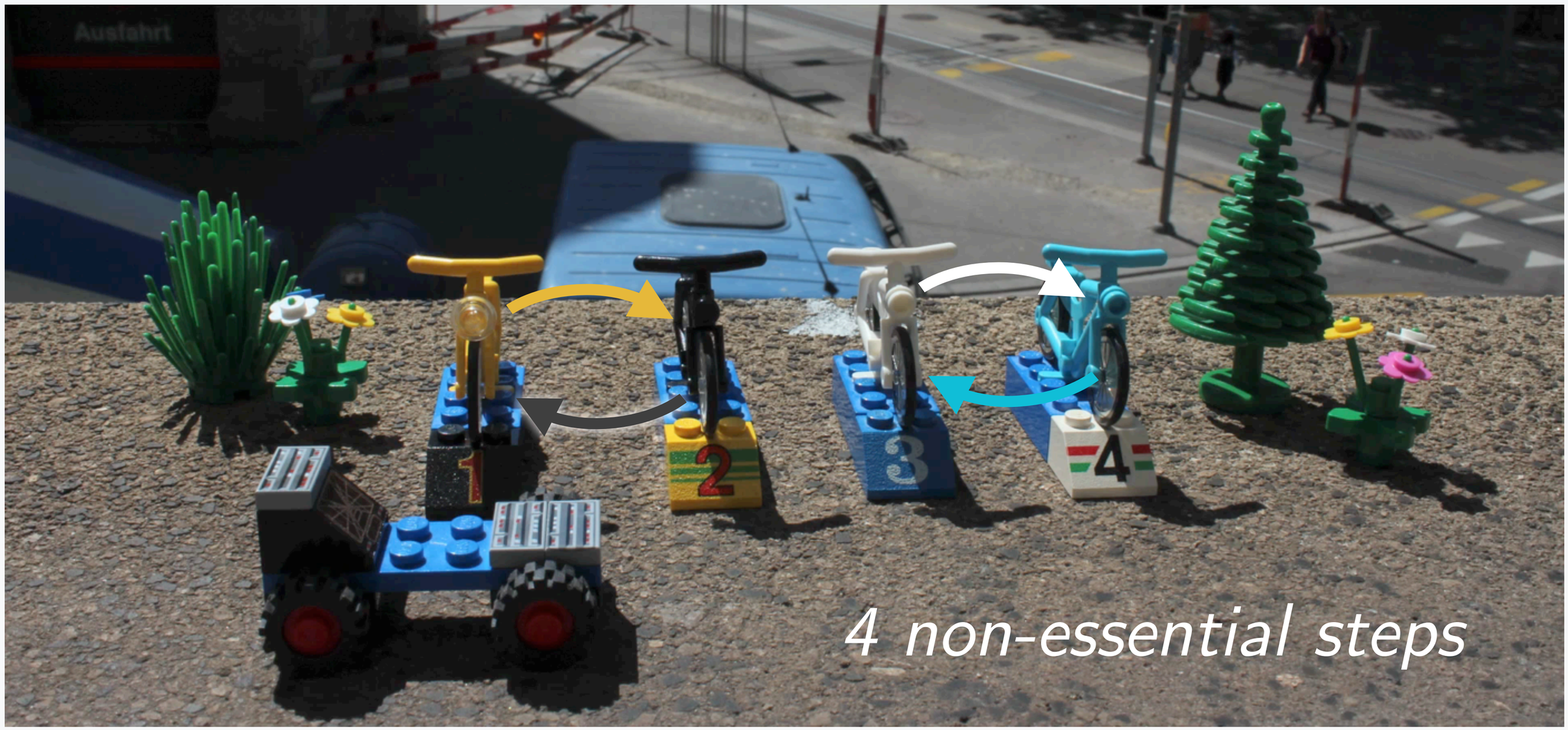

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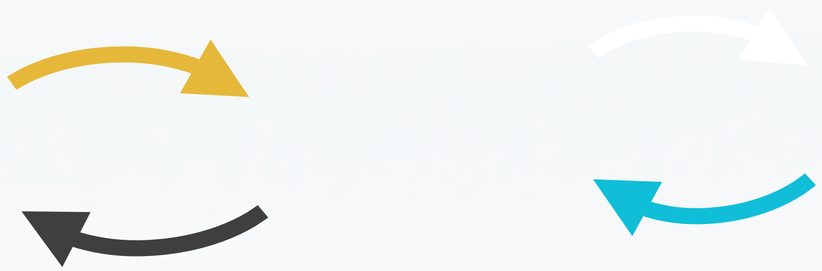

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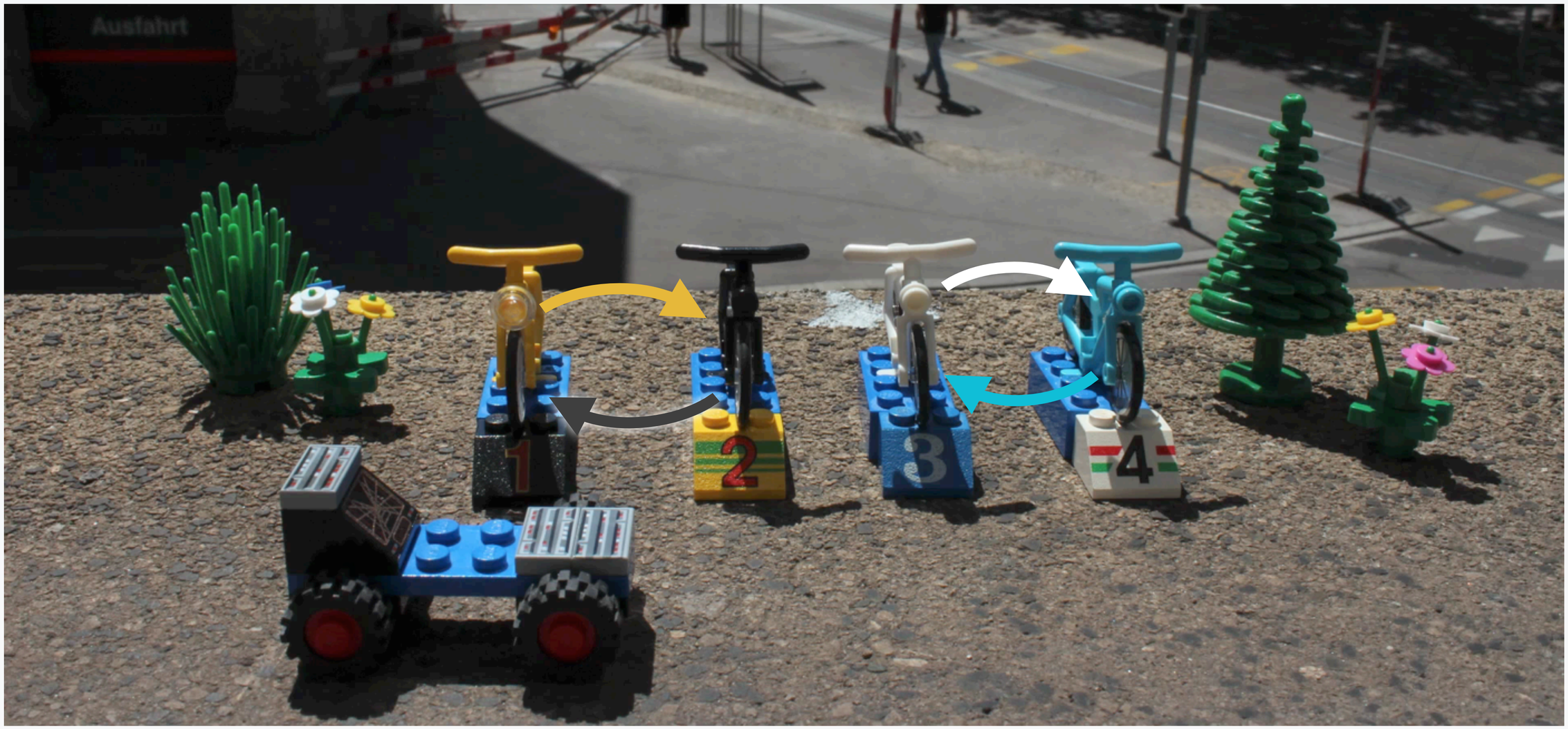

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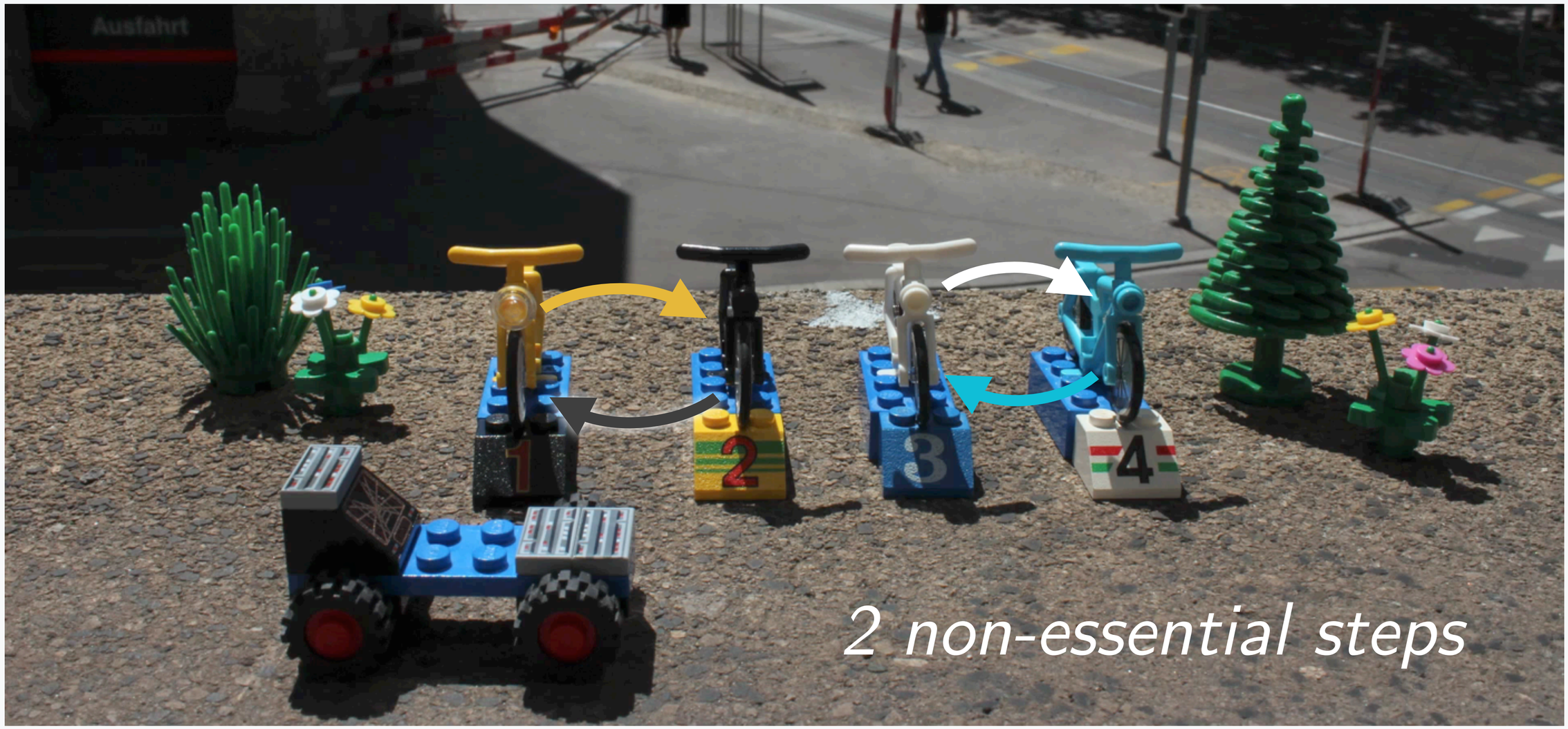

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# On Paths



*Theorem:* The shortest sorting path on a path $P$ has $d(\pi)+2\cdot(\#\textit{uncovered edges between cycles})$ steps and can be found in time $\mathcal{O}(n^2)$.

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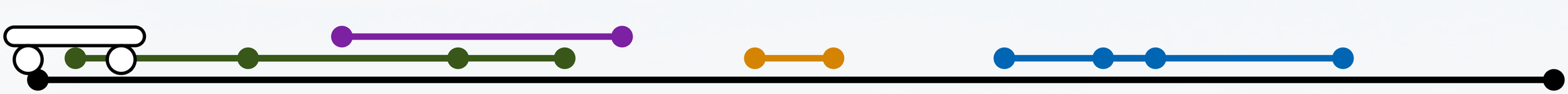

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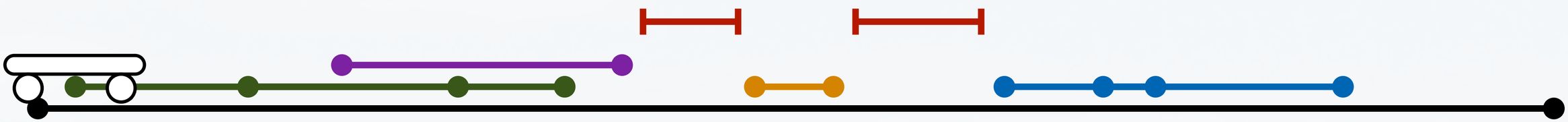

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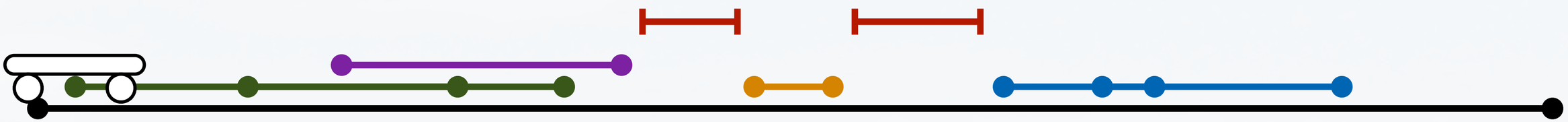

*Proof:* Induction over the cycles of $\pi$.

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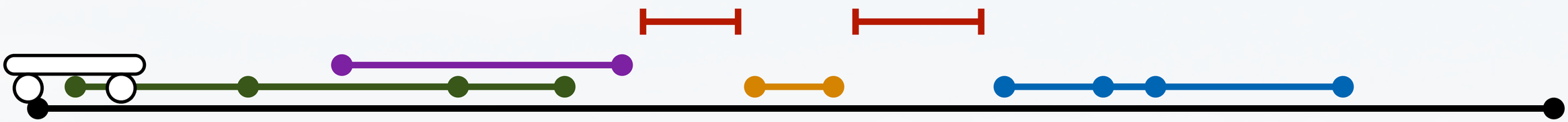
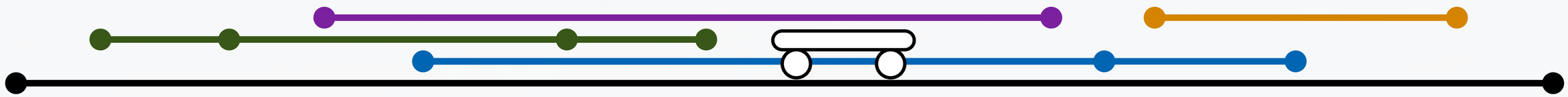


*Proof:* Induction over the cycles of $\pi$.

*Open:* What if we do not start at the border?



Non-essential steps needed also inside the cycles.

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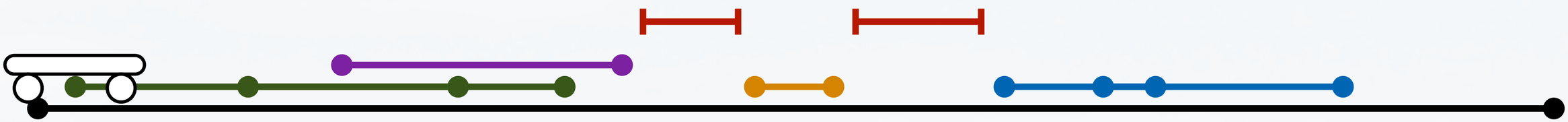

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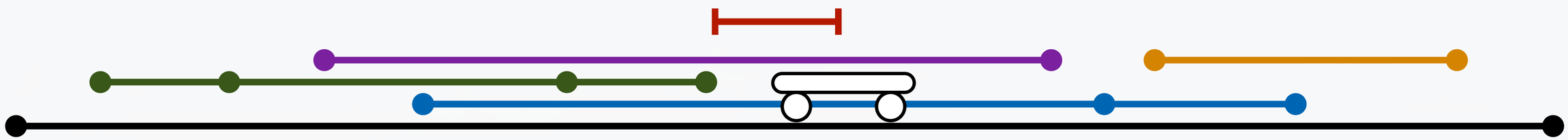
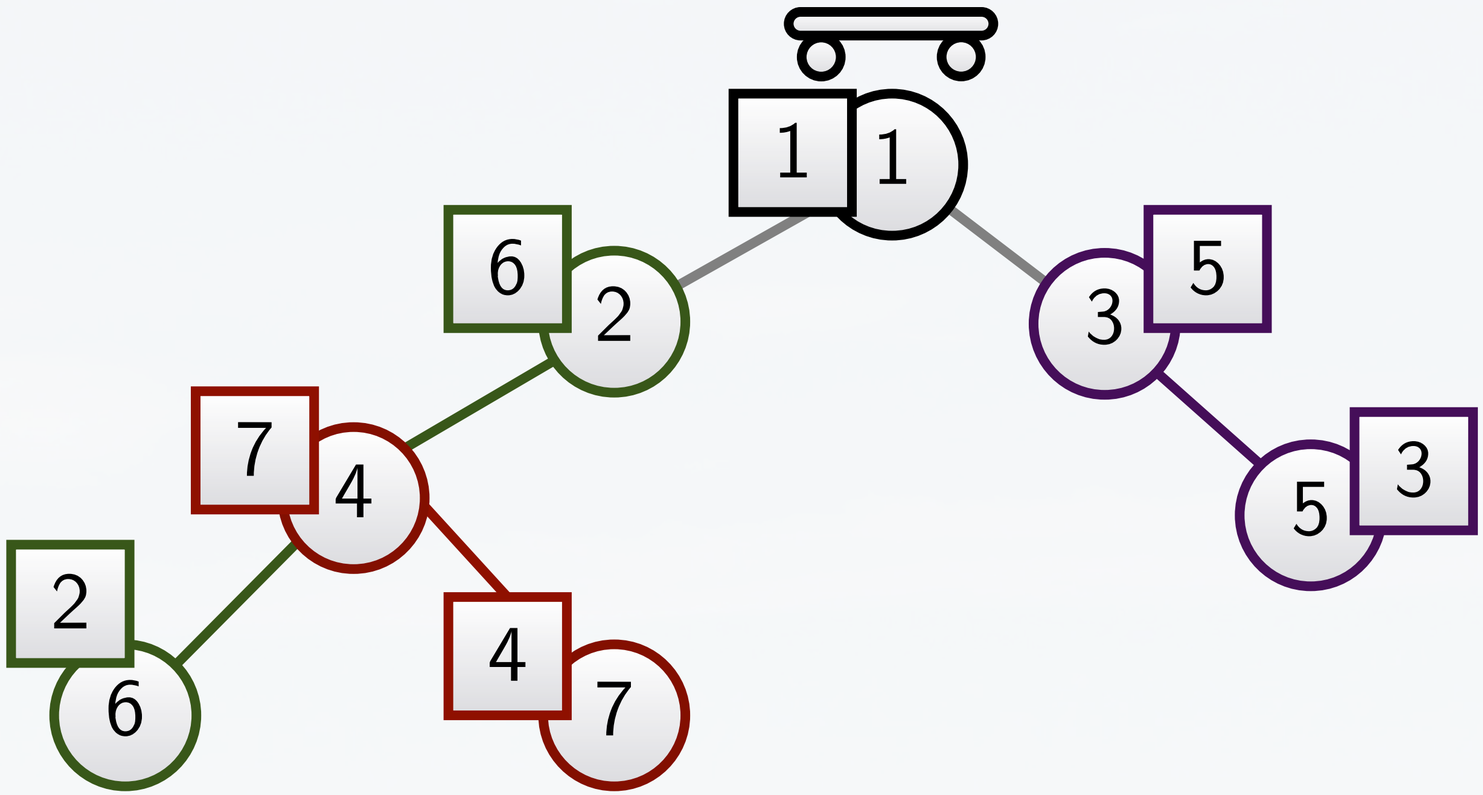


Non-essential steps needed also inside the cycles.

# On Trees



How do we minimize the non-essential steps?



# On Trees



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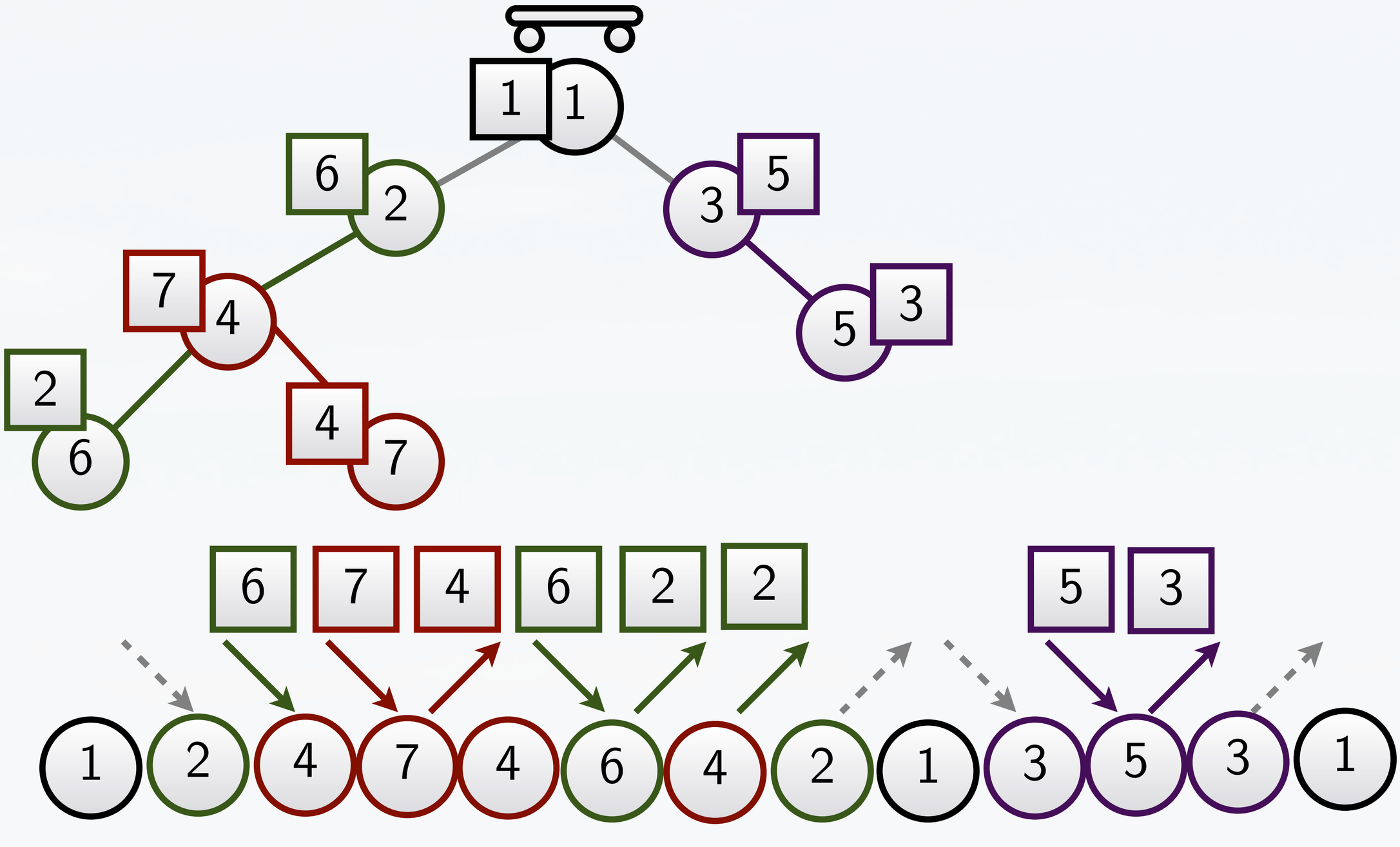
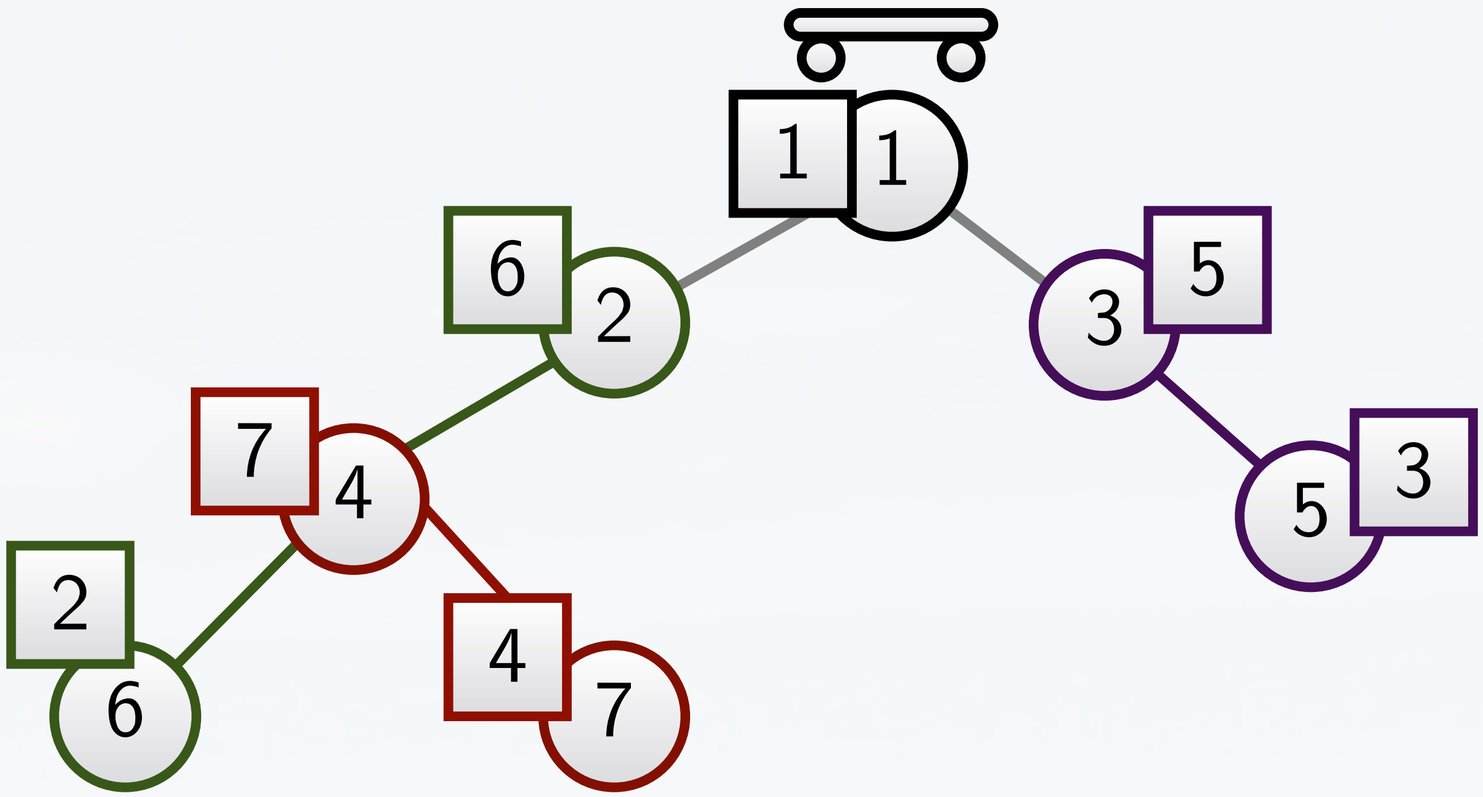


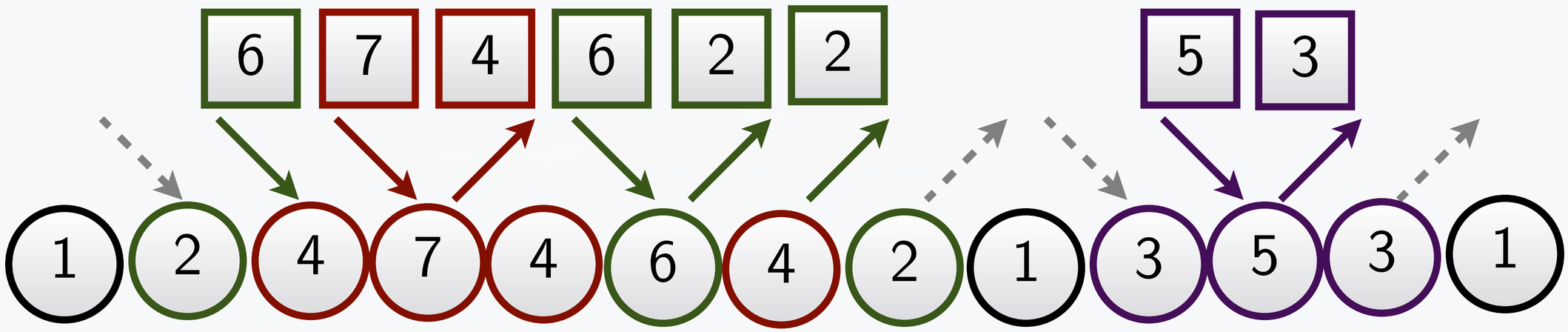
# On Trees



How do we minimize the non-essential steps?

cycle anchor tree



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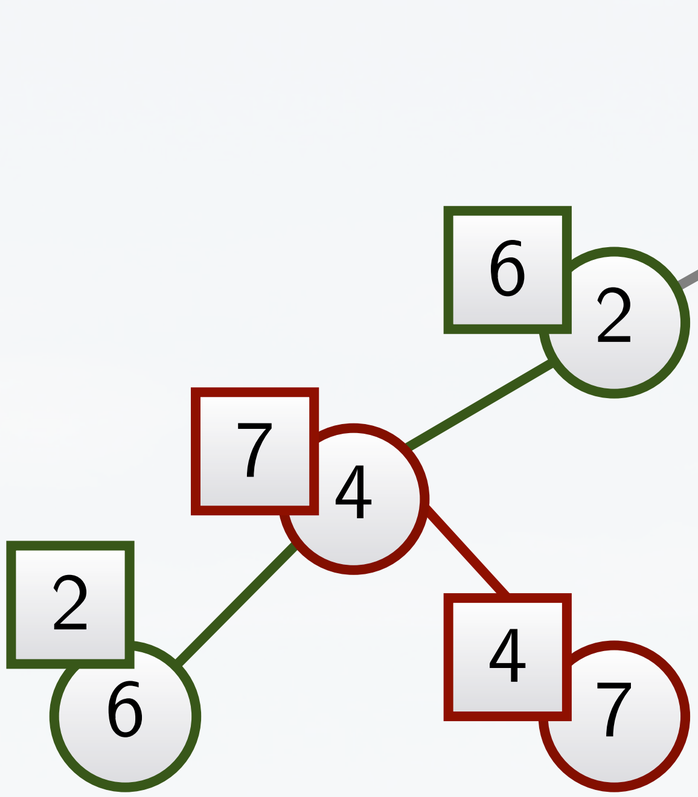


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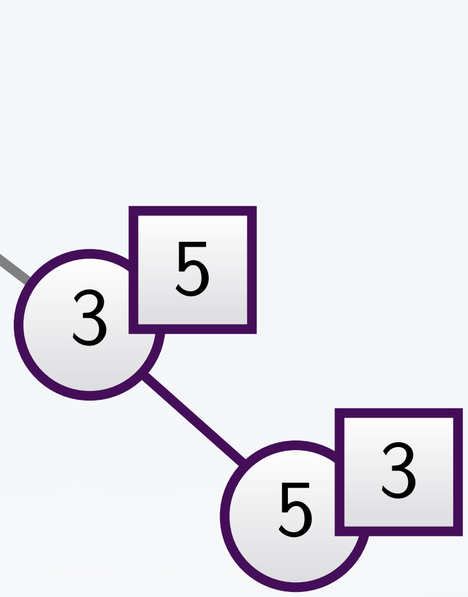


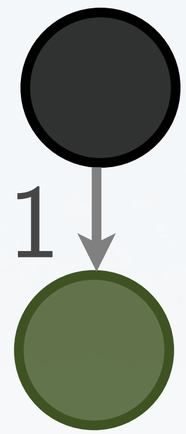


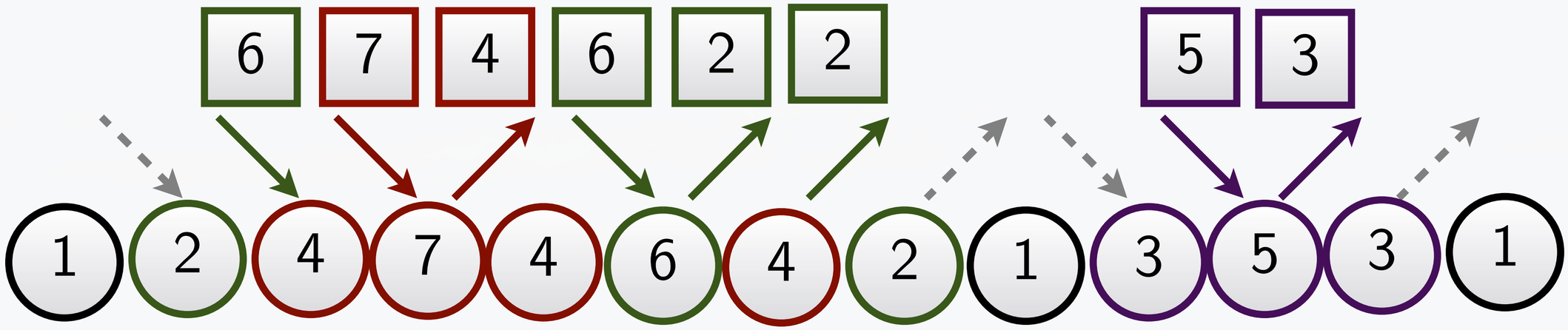

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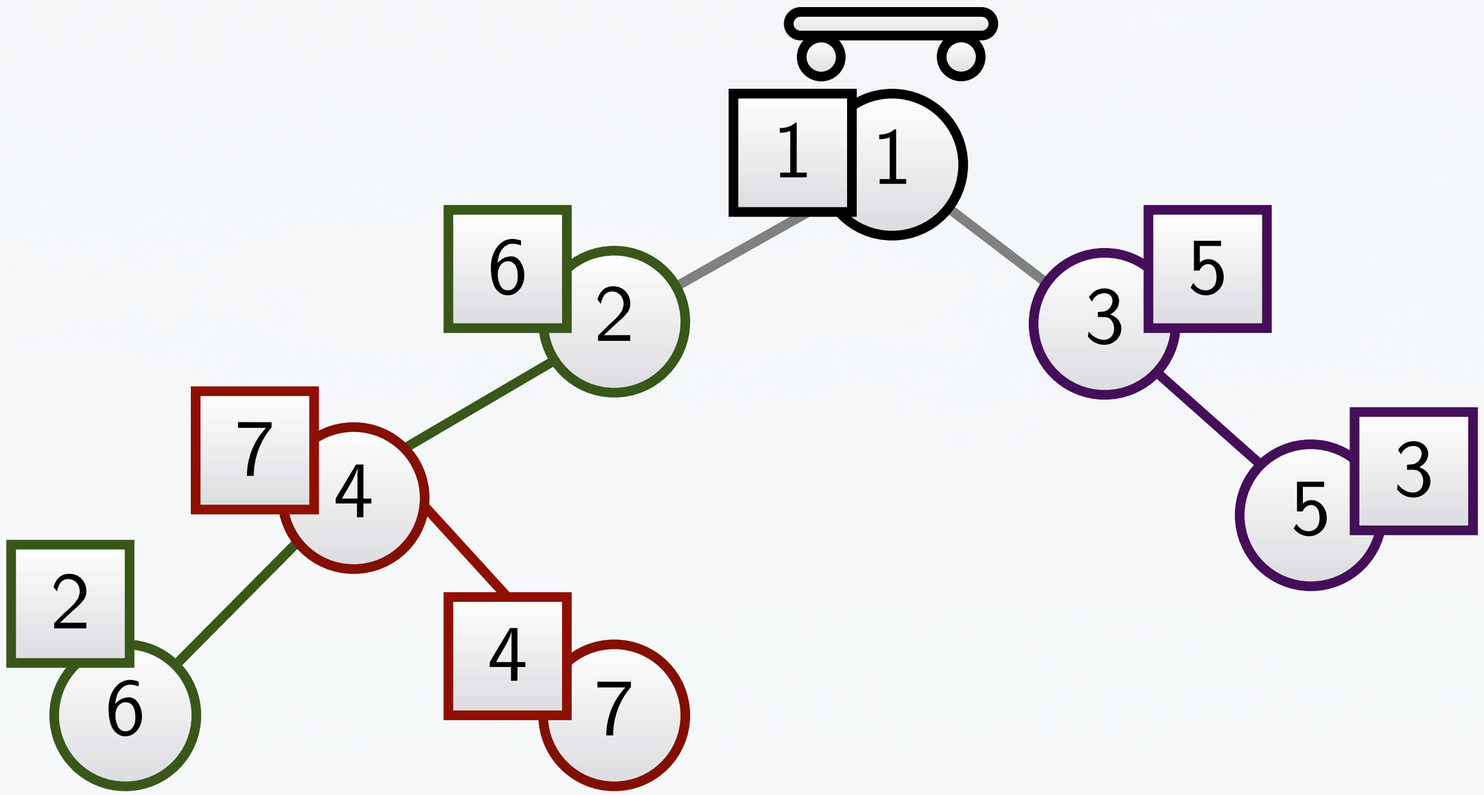


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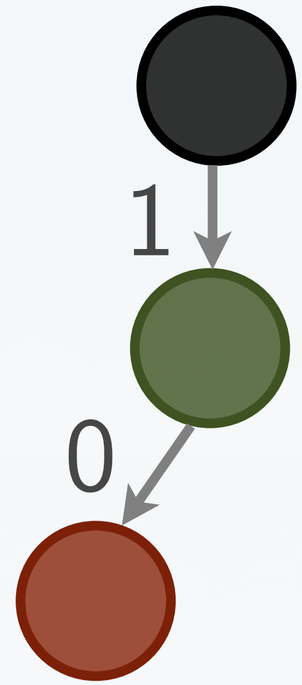


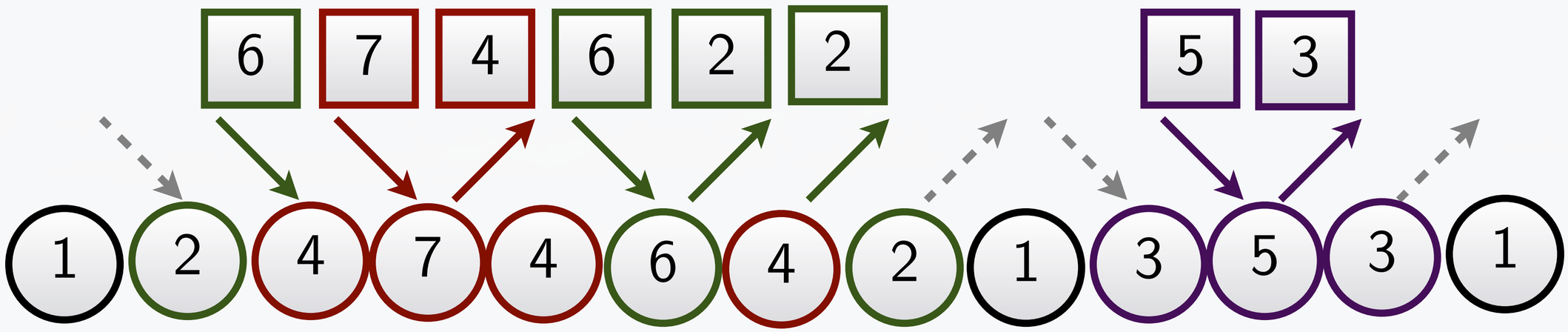

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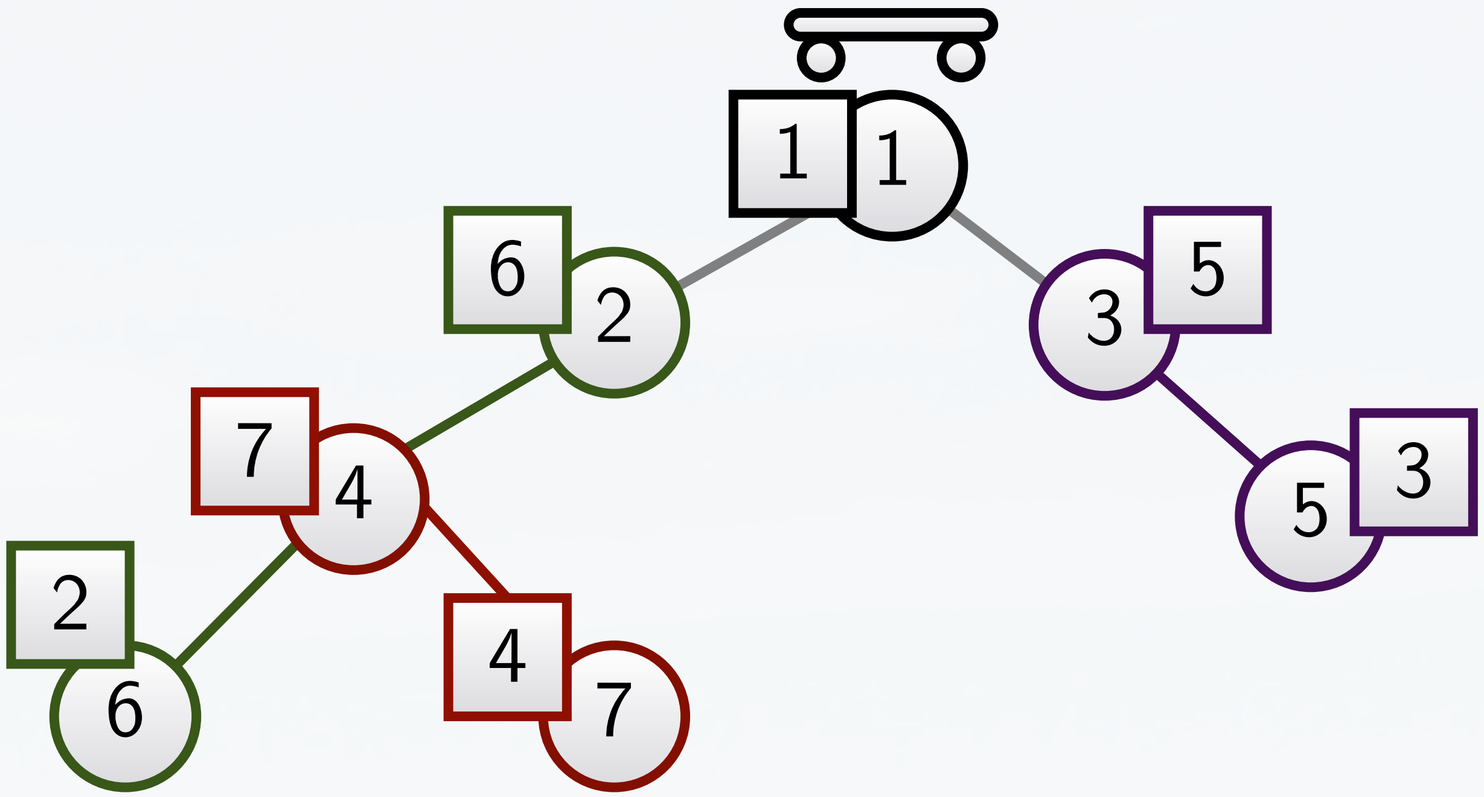


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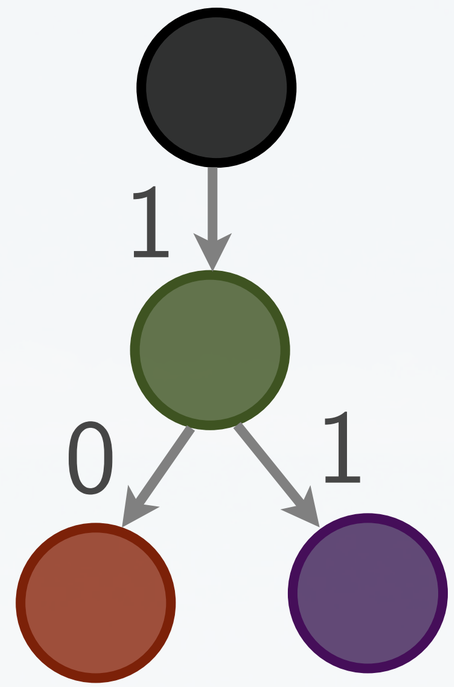
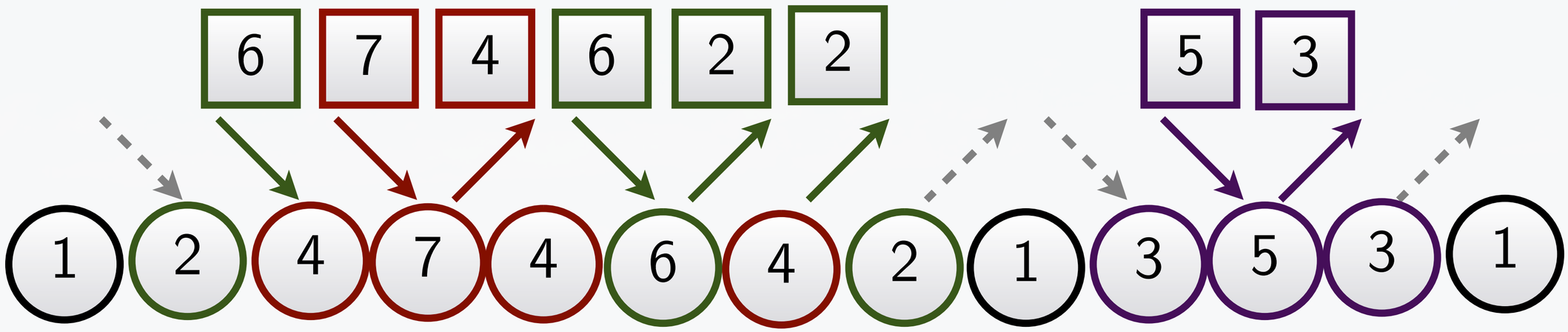


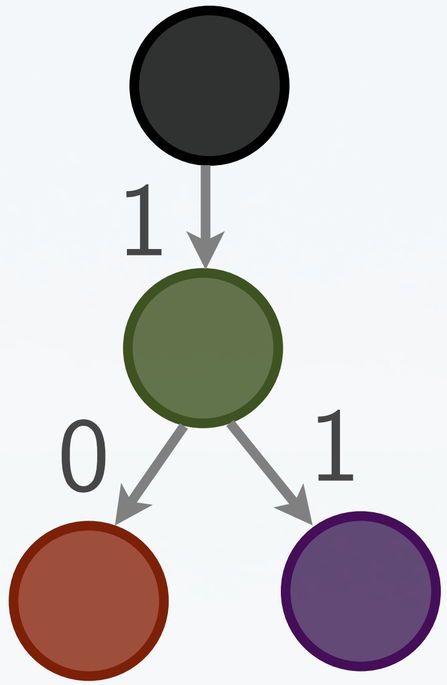
# On Trees



How do we minimize the non-essential steps?

*Edge cost* $c((C_1, C_2)) =$ minimal number of down steps from any vertex covered by $C_1$ to a vertex containing a box of $C_2$

cycle anchor tree



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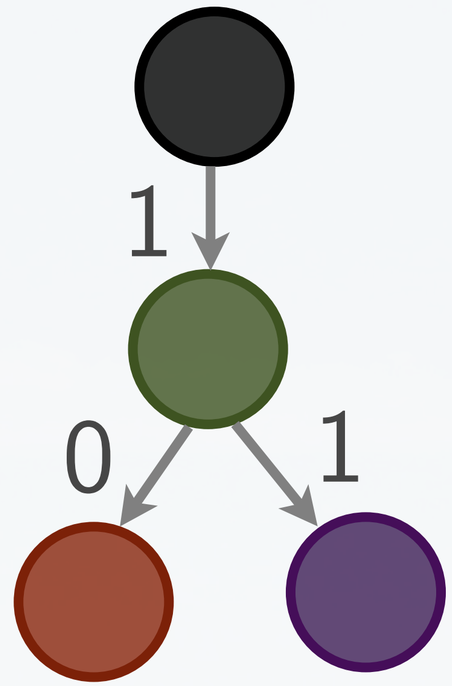


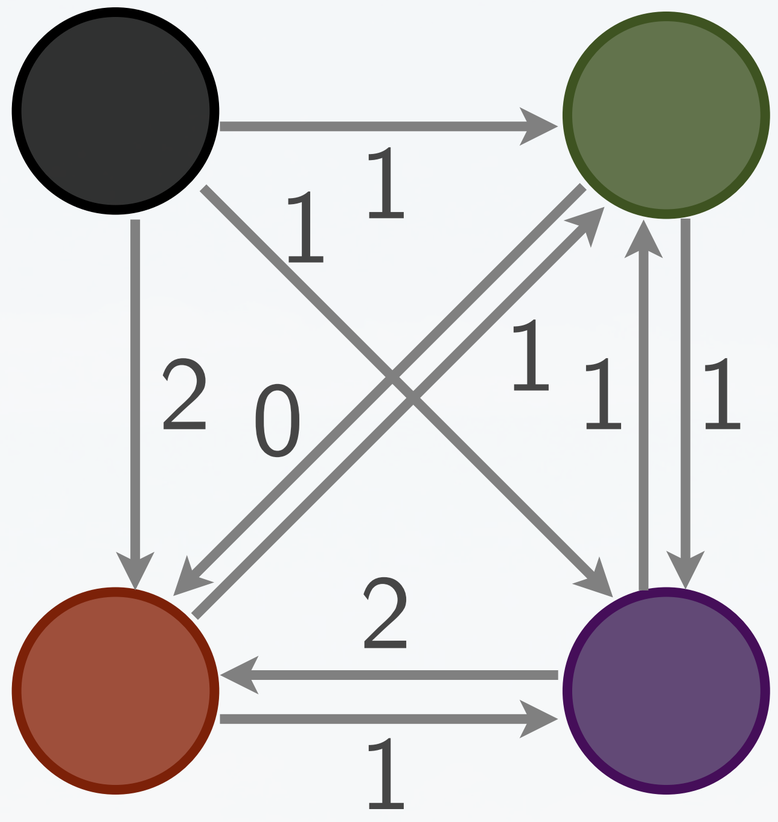

*Observation:* A cycle anchor tree determines the length of all corresponding sorting walks.

# On Trees



How do we find the cheapest cycle anchor tree?

- Start with the complete directed graph on all cycles.
- Find a cheapest direct subgraph rooted at ● using Edmond's algorithm for optimum branchings.



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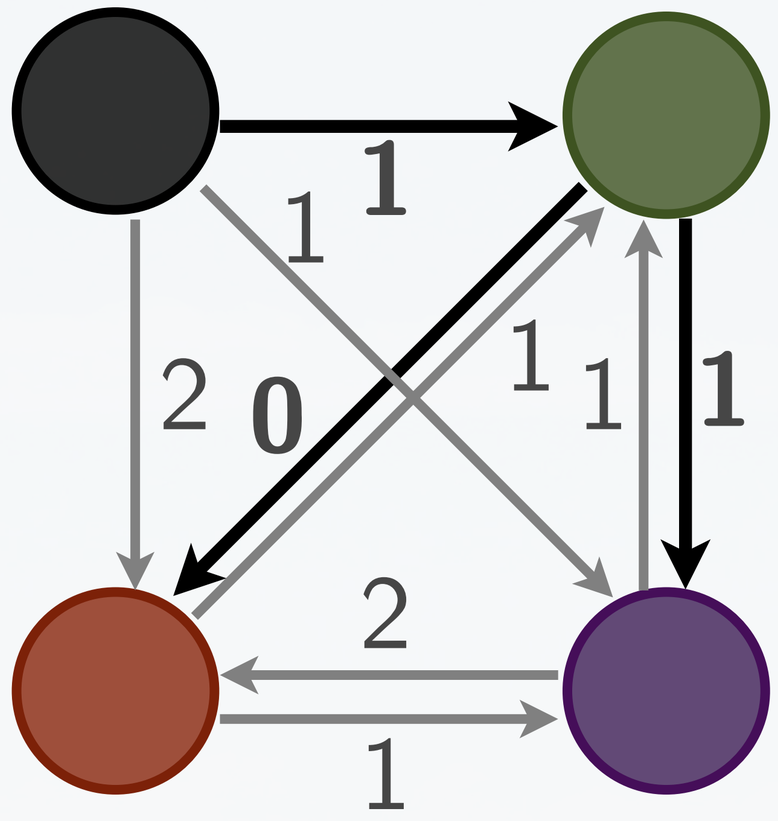

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# On Trees



*Theorem:* The shortest sorting path on a tree $T$ can be found in time $\mathcal{O}(n^2)$.

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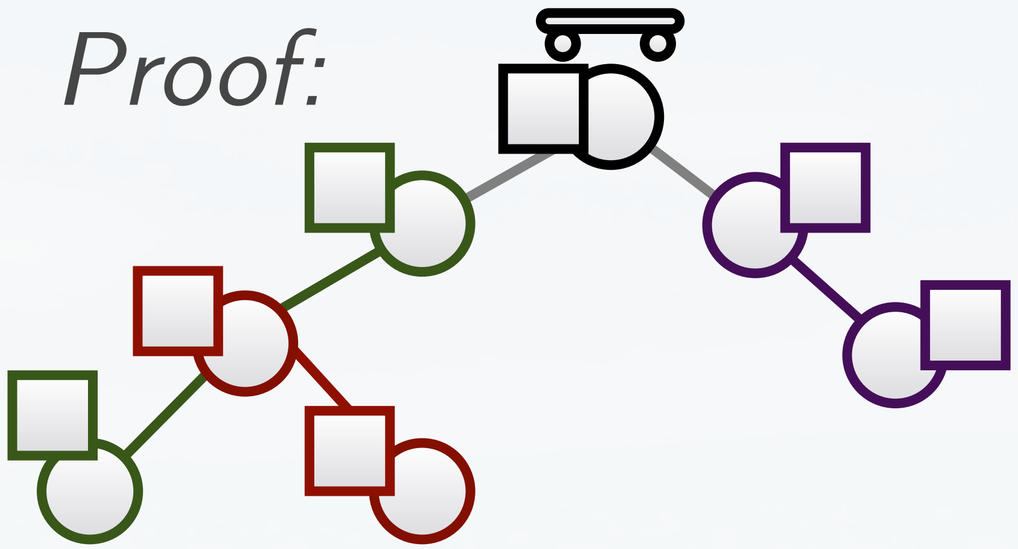
*Theorem:* The shortest sorting path on a tree $T$ can be found in time $\mathcal{O}(n^2)$.

*Proof:*



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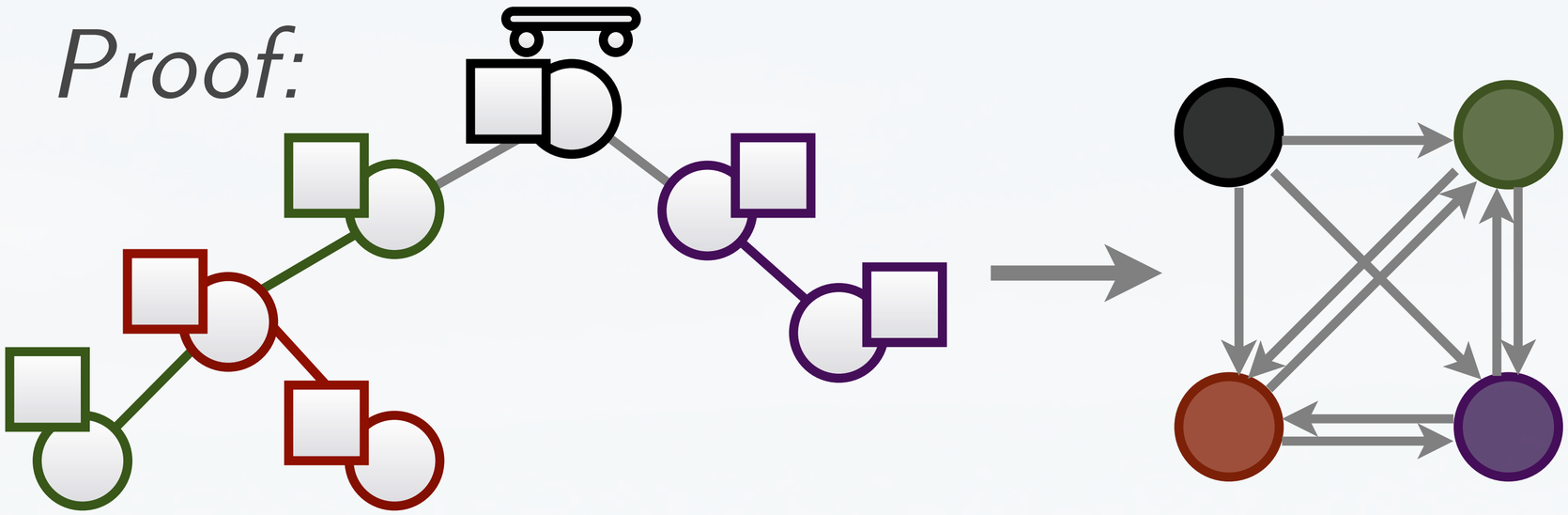

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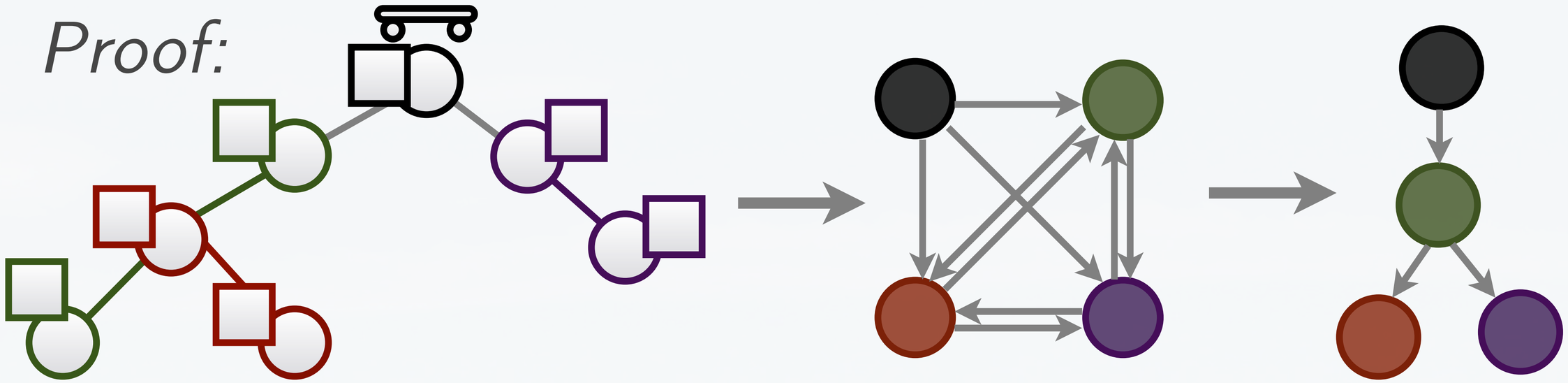

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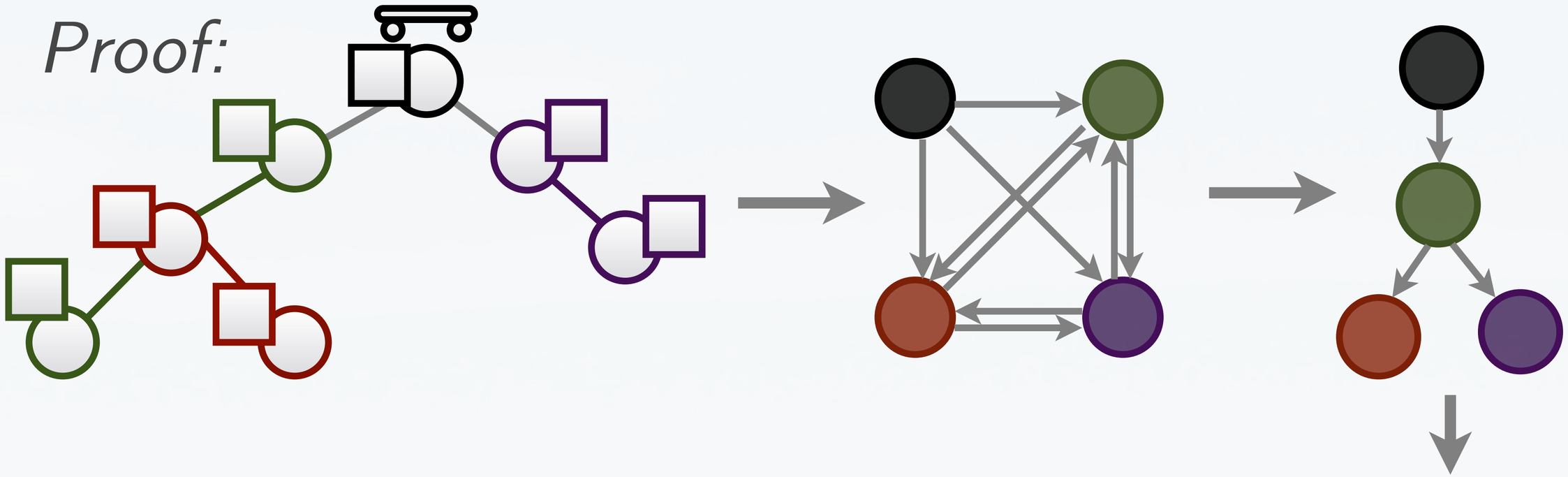

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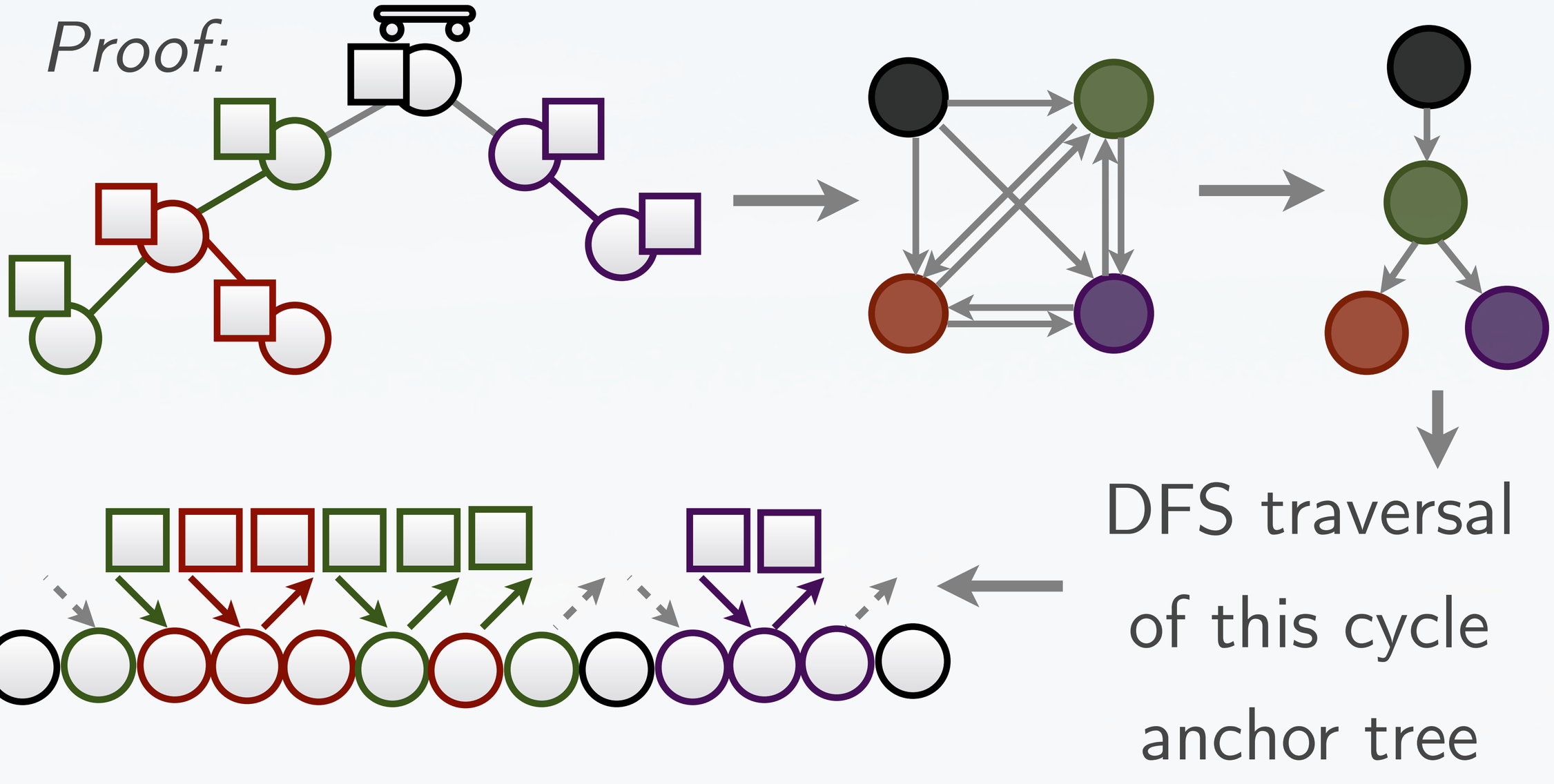

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# On Circles



*Theorem:* The shortest sorting path on a circle $C$ can be found in time $\mathcal{O}(n^3)$.

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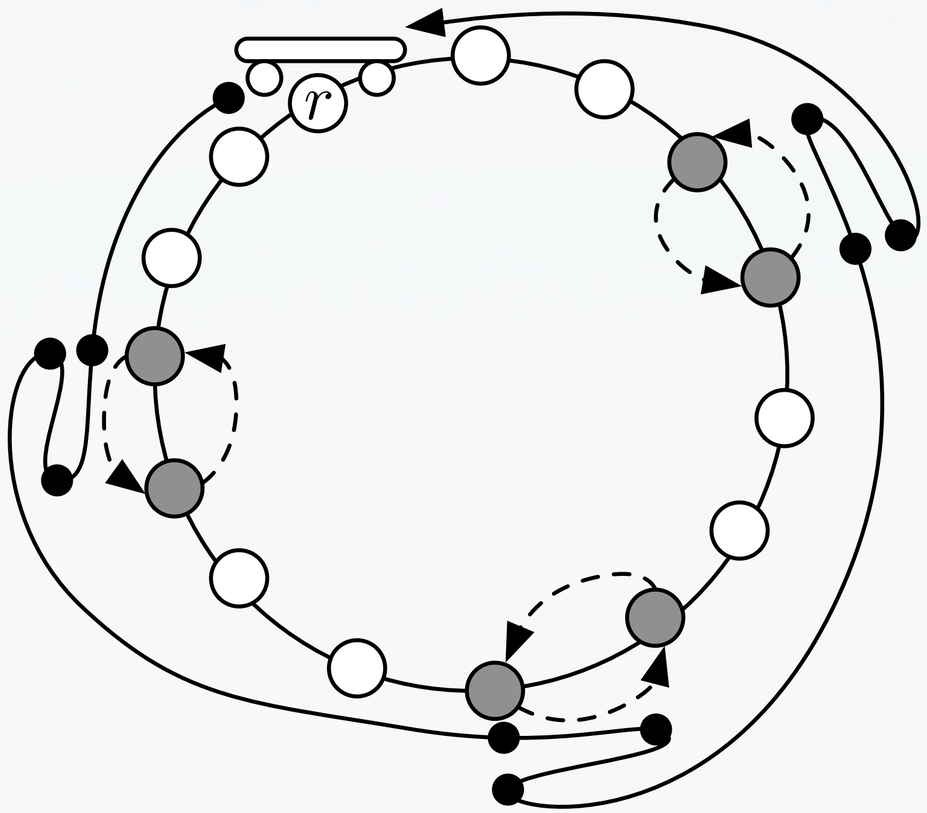
# On Circles



*Theorem:* The shortest sorting path on a circle $C$ can be found in time $\mathcal{O}(n^3)$.

*Proof Ideas:* Full circle might be necessary.



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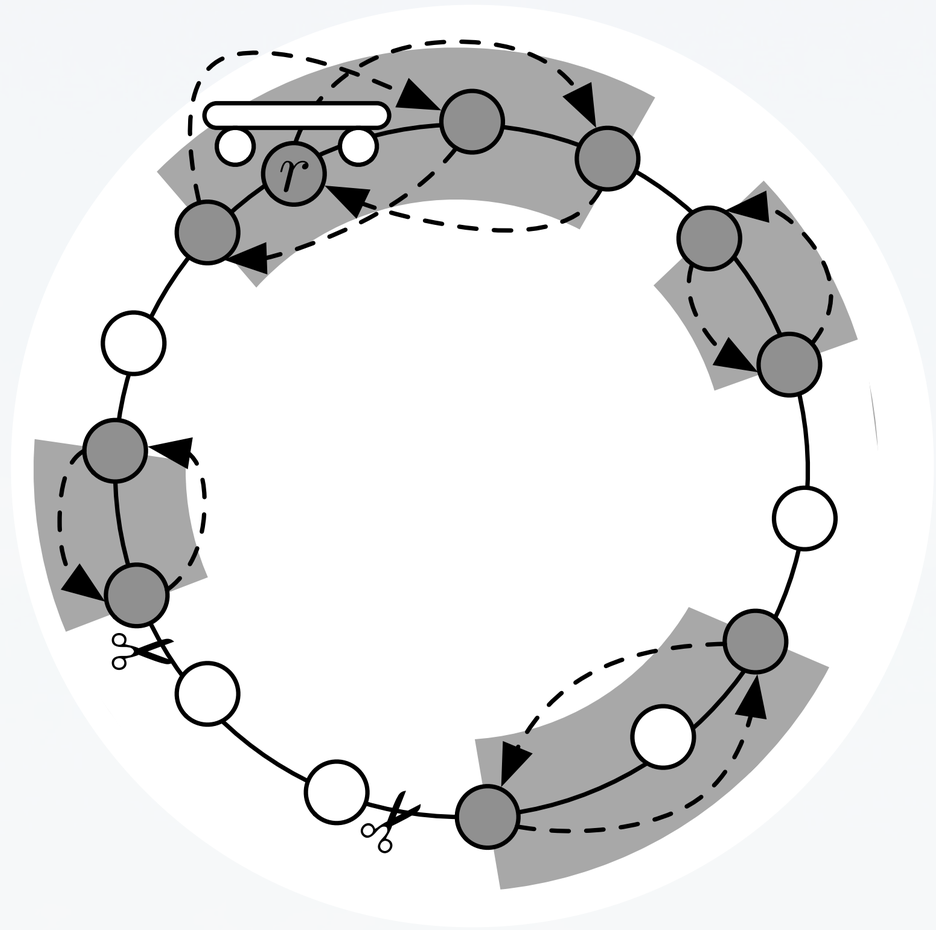
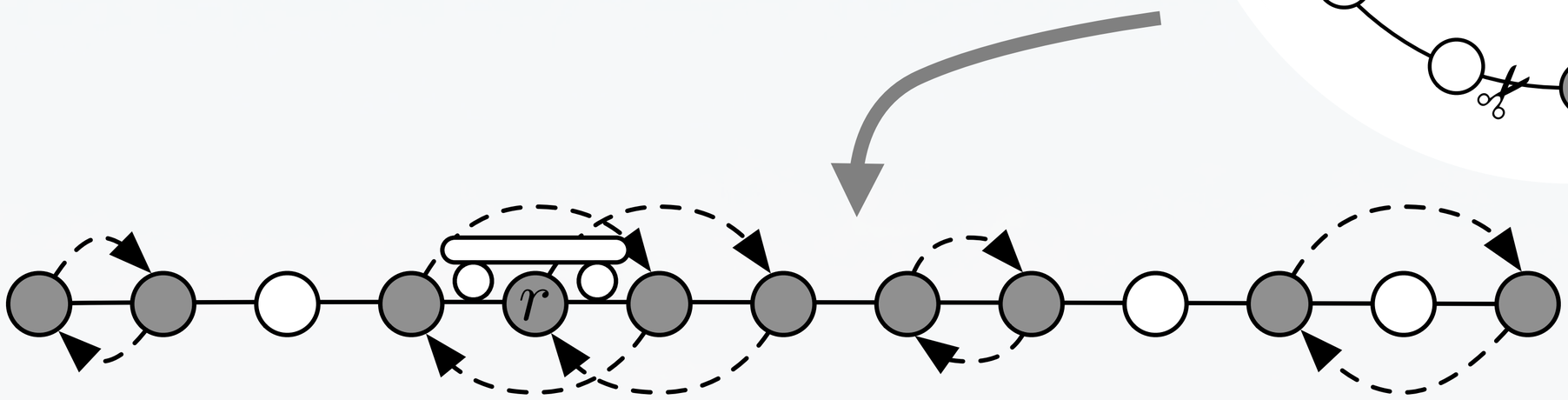
# On Circles



*Theorem:* The shortest sorting path on a circle $C$ can be found in time $\mathcal{O}(n^3)$.

*Proof Ideas:* Reduce it to sorting on a path.



# Planar Graphs



What if we look at more general graphs and allow multiple cycles?

# Planar Graphs



What if we look at more general graphs and allow multiple cycles?

*Theorem:* Finding a shortest sorting walk is *NP*-complete for planar graphs.

# Planar Graphs



What if we look at more general graphs and allow multiple cycles?

*Theorem:* Finding a shortest sorting walk is *NP*-complete for planar graphs.

*Proof:* Reduction from the problem of finding Hamiltonian circuits for grid graphs.

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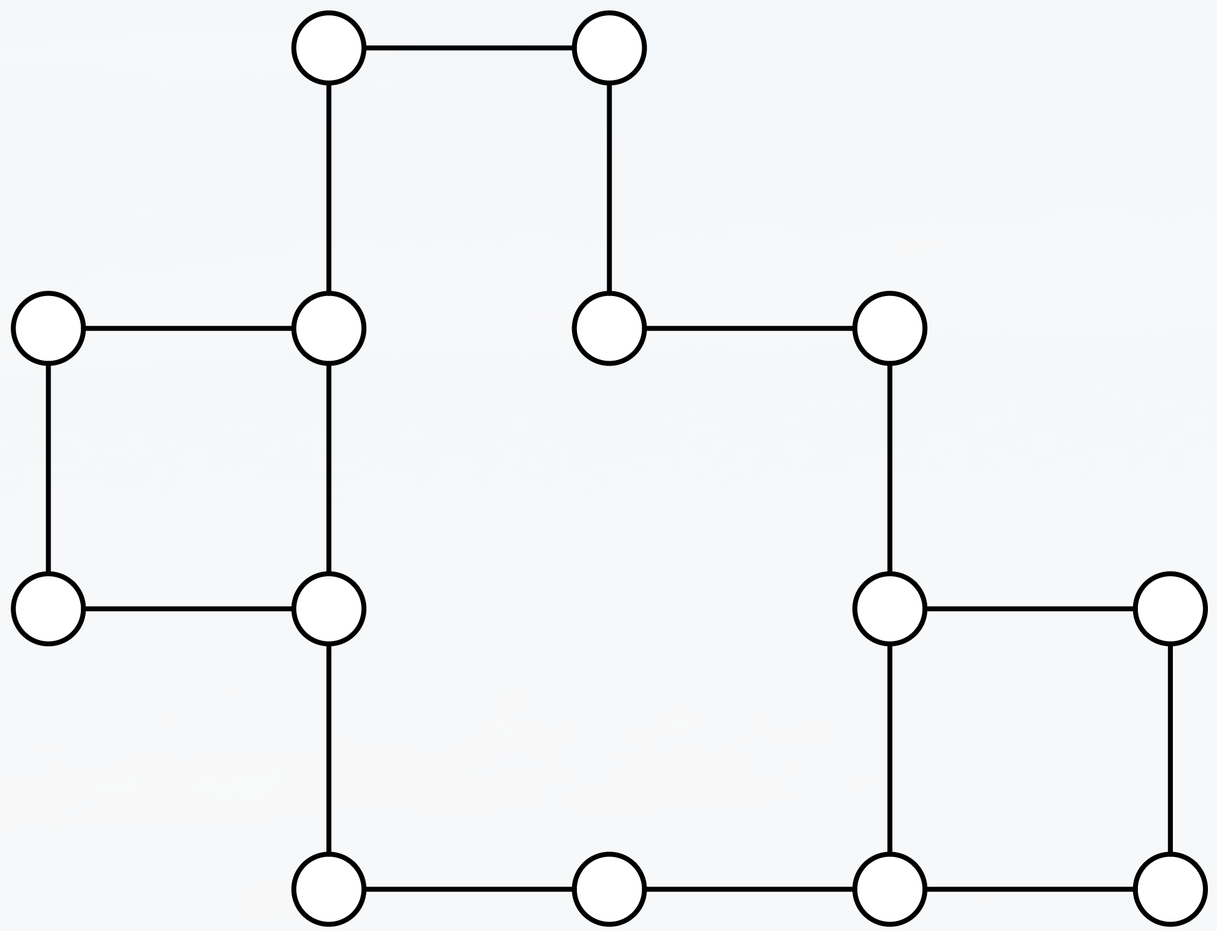
# Planar Graphs



*Theorem [Itai et al. 1981]:* Finding Hamiltonian circuits for grid graphs is *NP*-complete.



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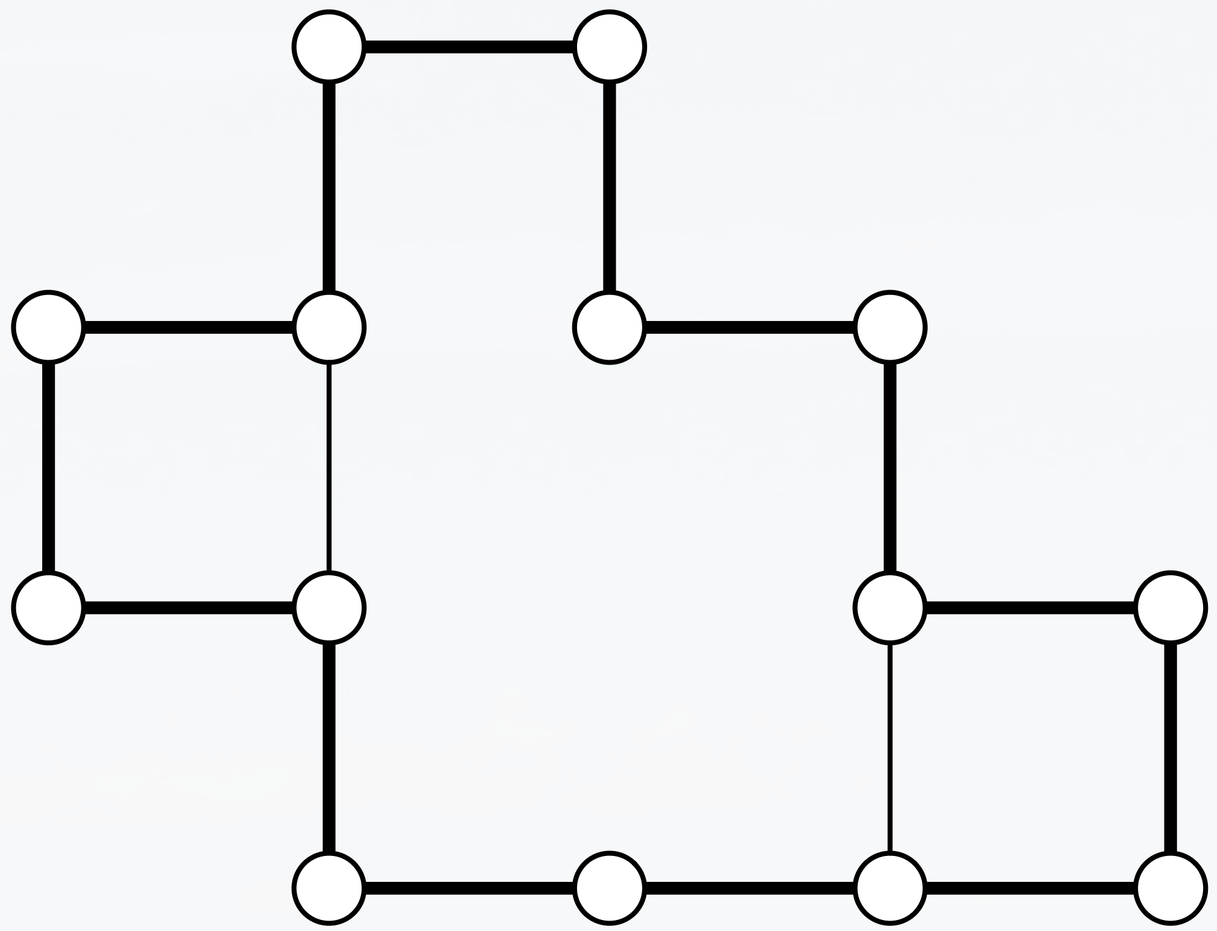

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# Planar Graphs



*Theorem:* Finding a shortest sorting walk is *NP*-complete for planar graphs.

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# Planar Graphs



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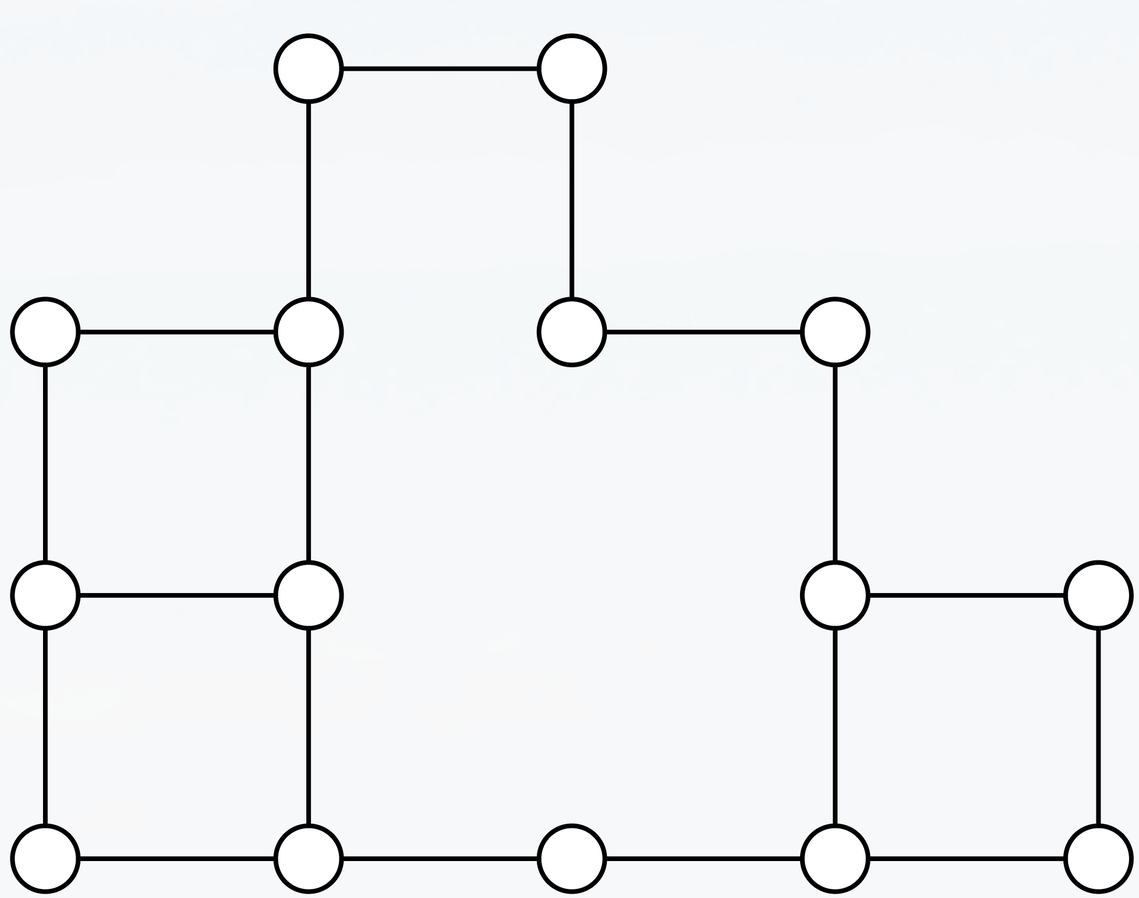

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# Planar Graphs



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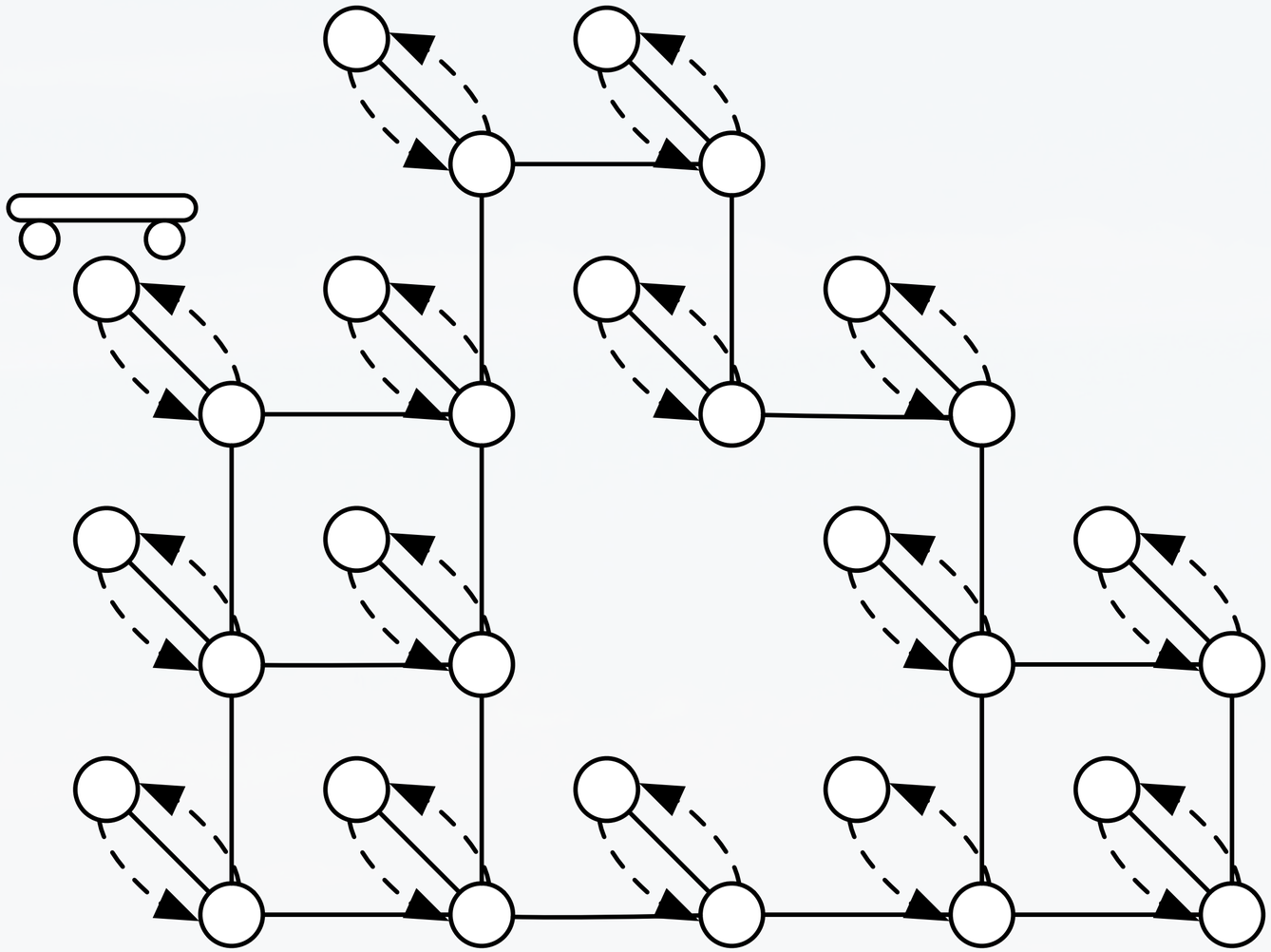

# Fewest Swaps



Box handling takes significant time in practice.

Different cost function:

- count number of swaps
- minimizing number of steps as a second priority

Observations:

- no interleaving of cycles anymore
- non-essential steps form a subtree containing $r$ and a vertex of every cycle of $\pi$.

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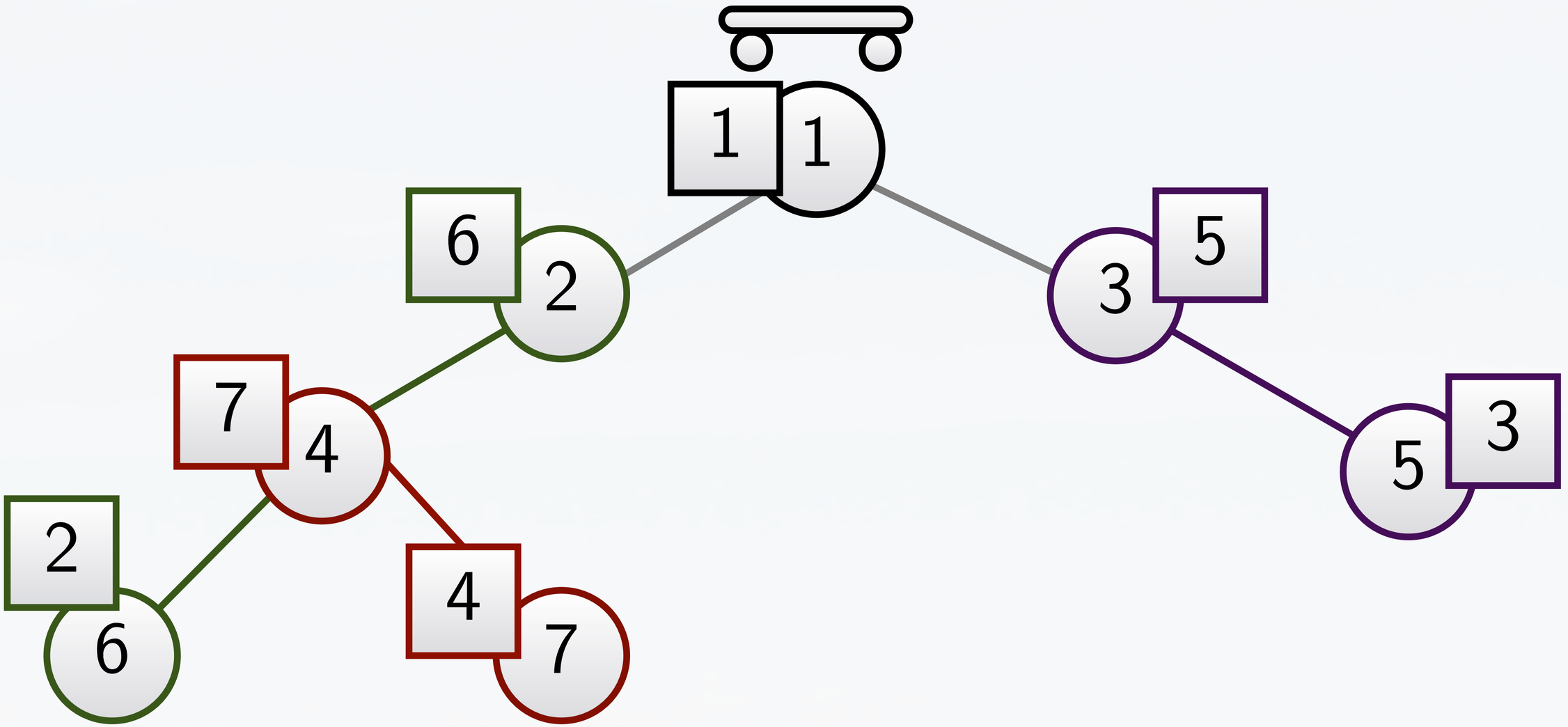
# Fewest Swaps





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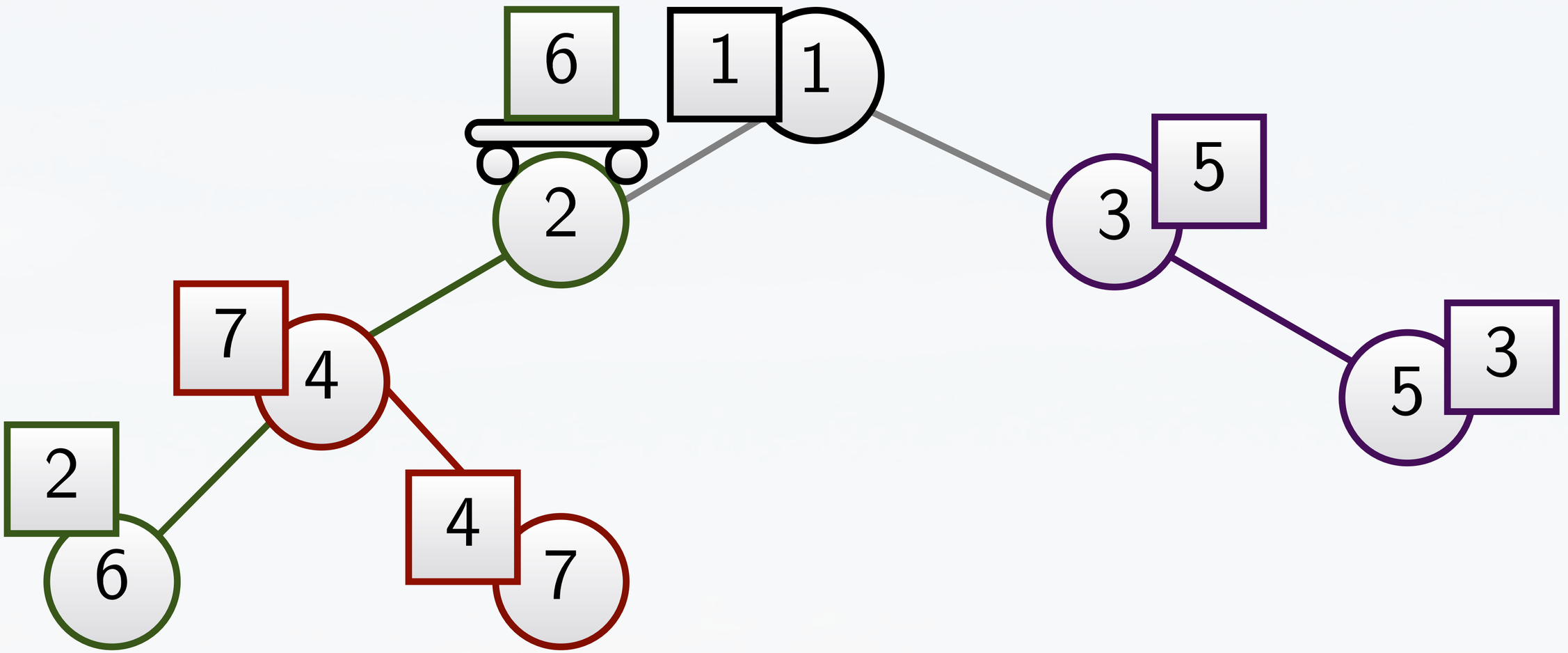

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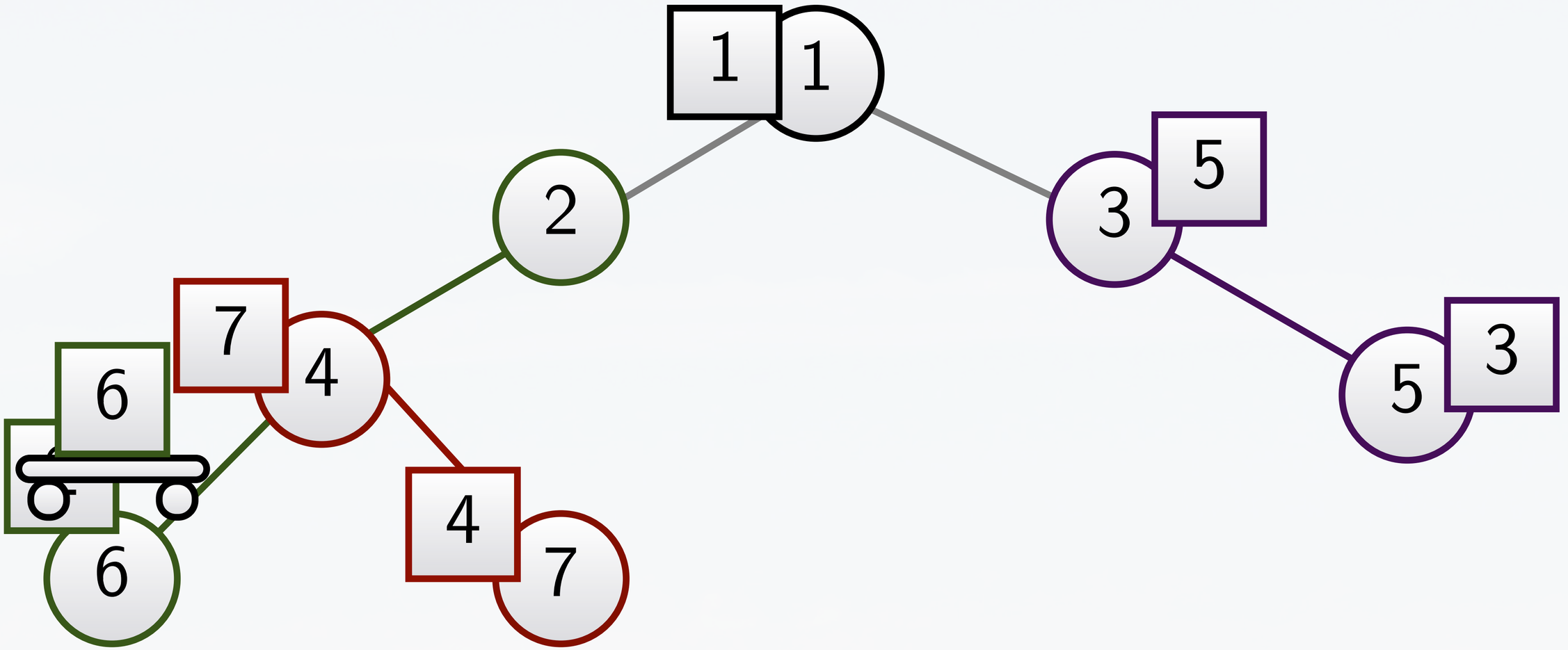

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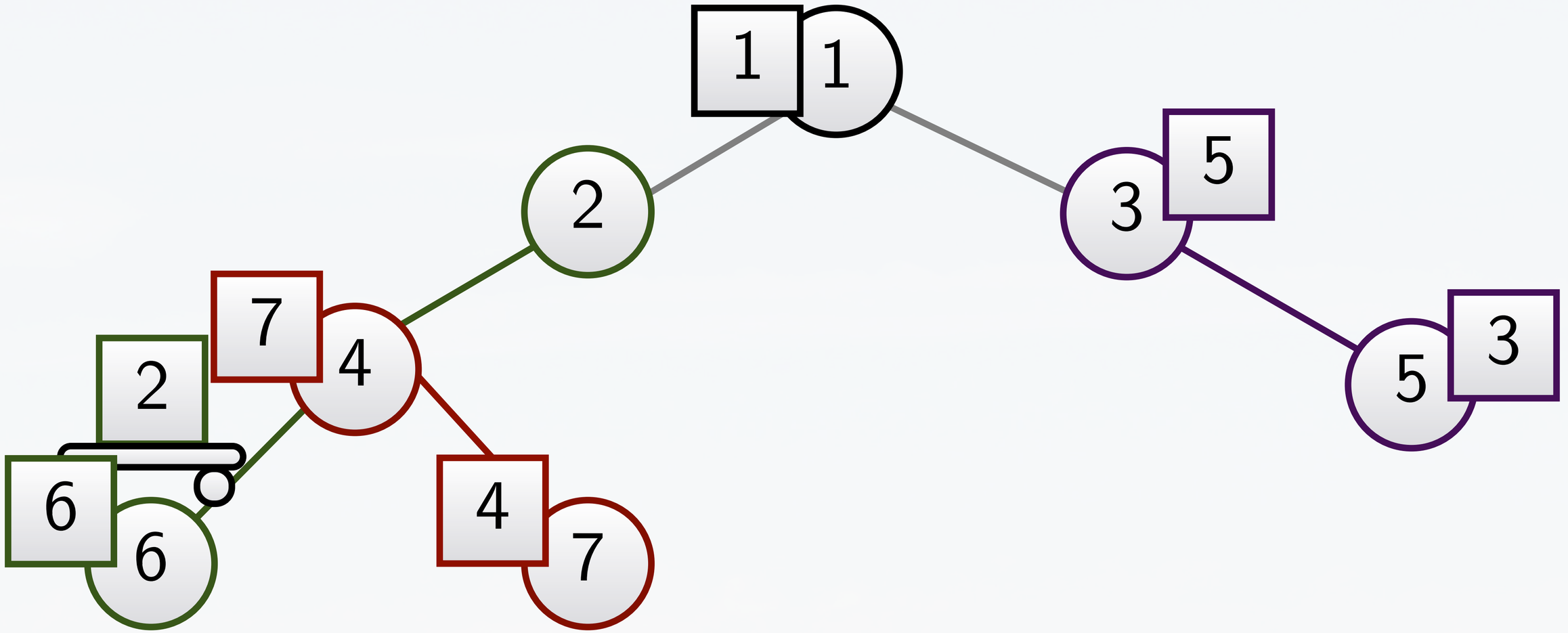

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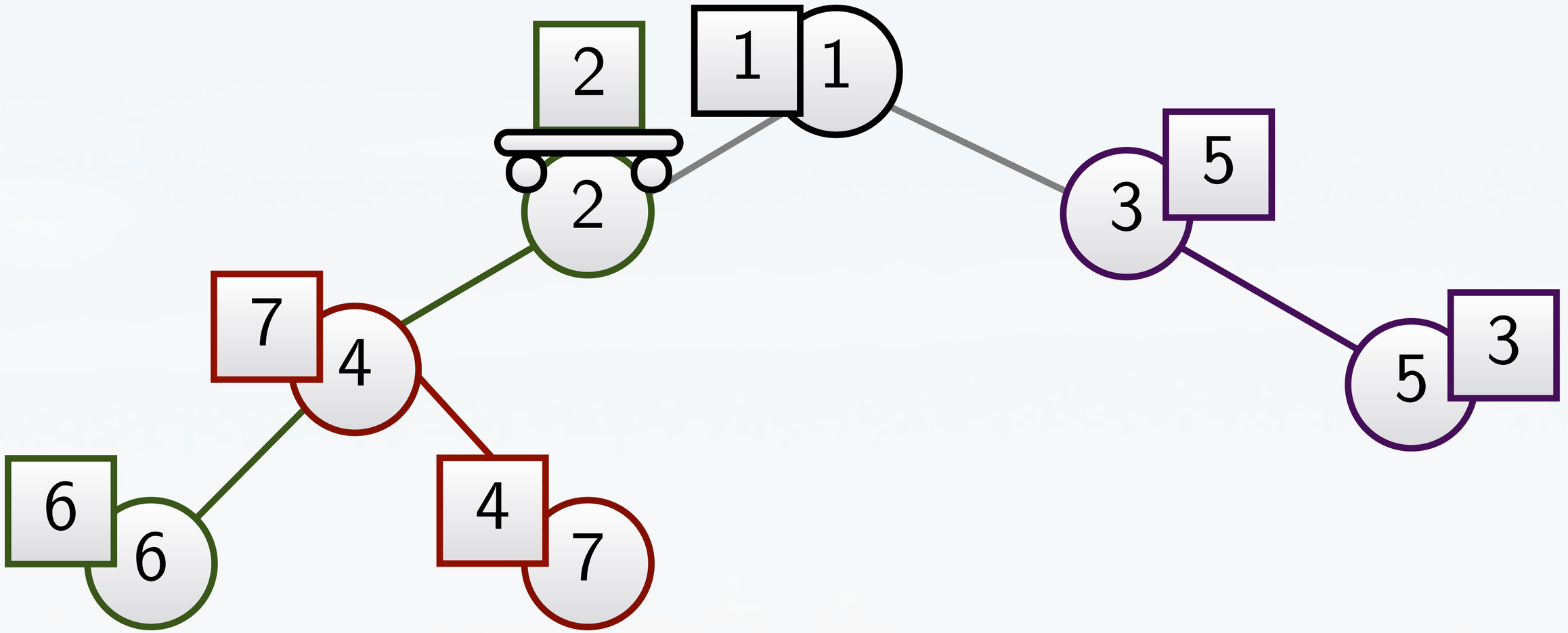

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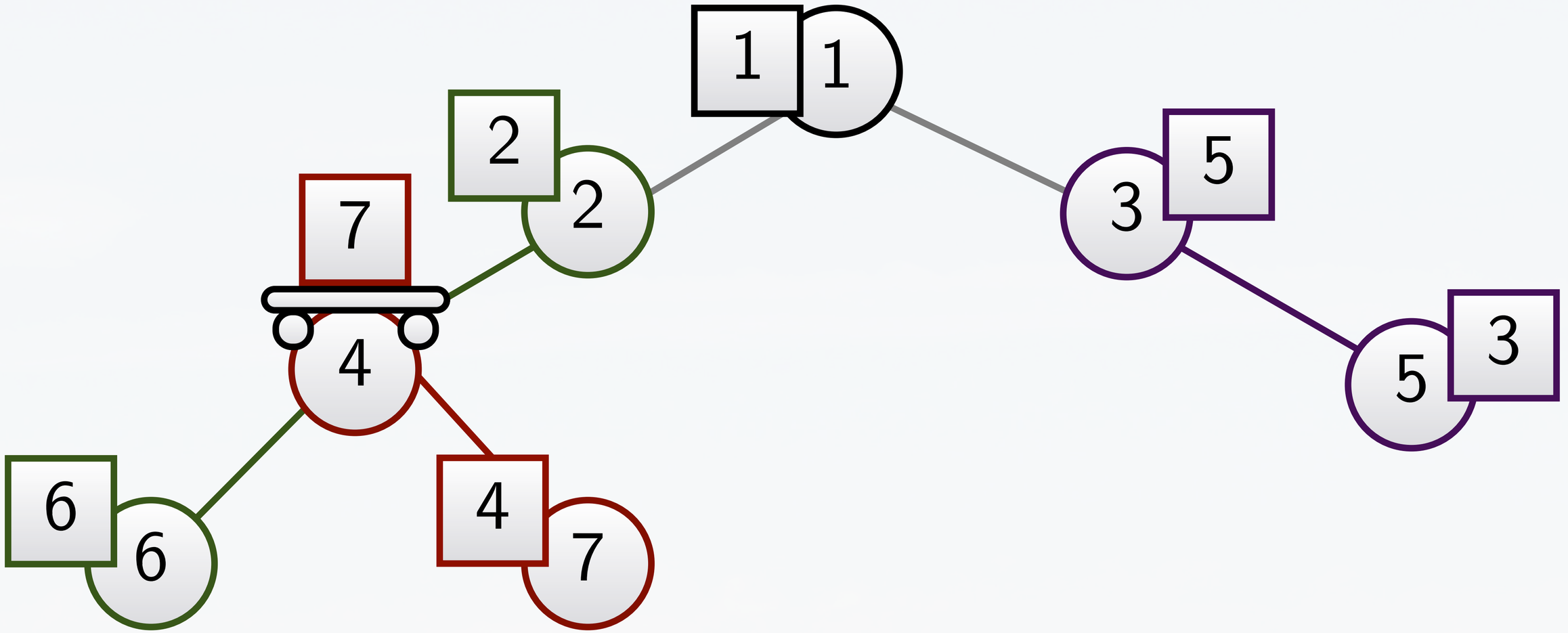

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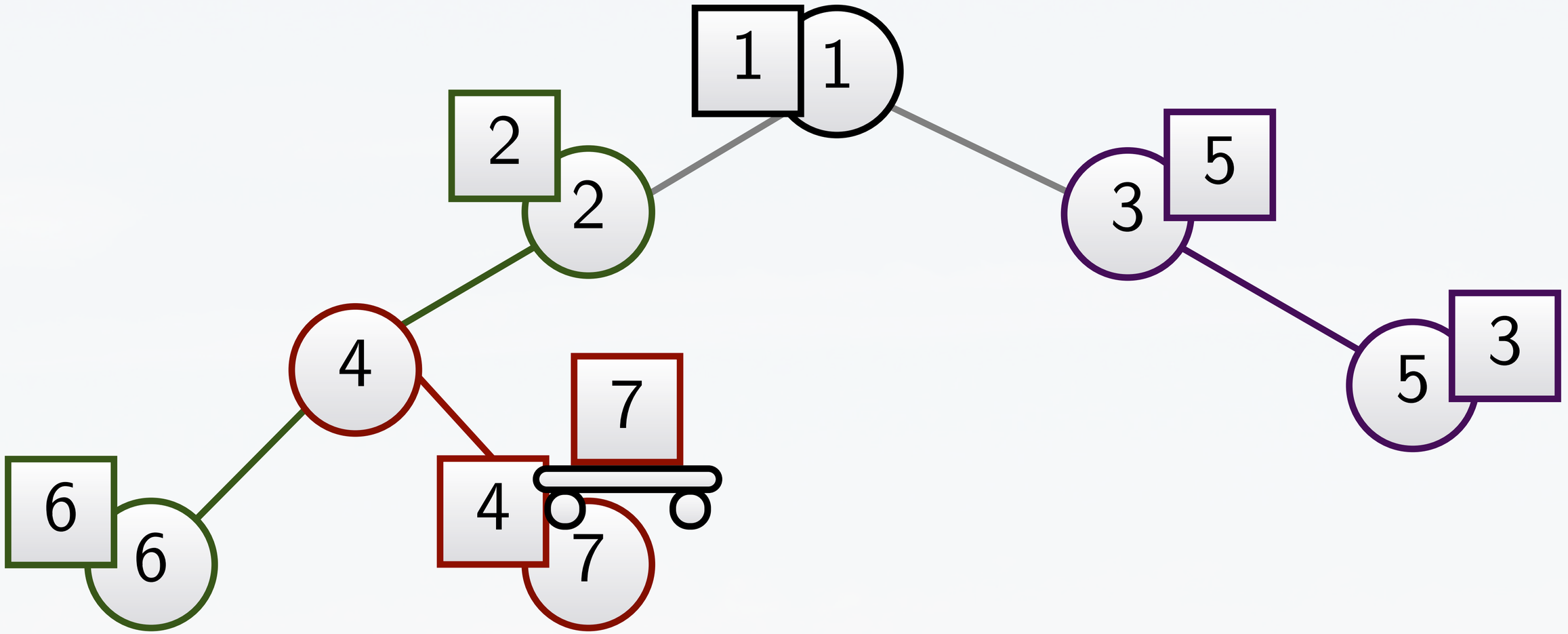

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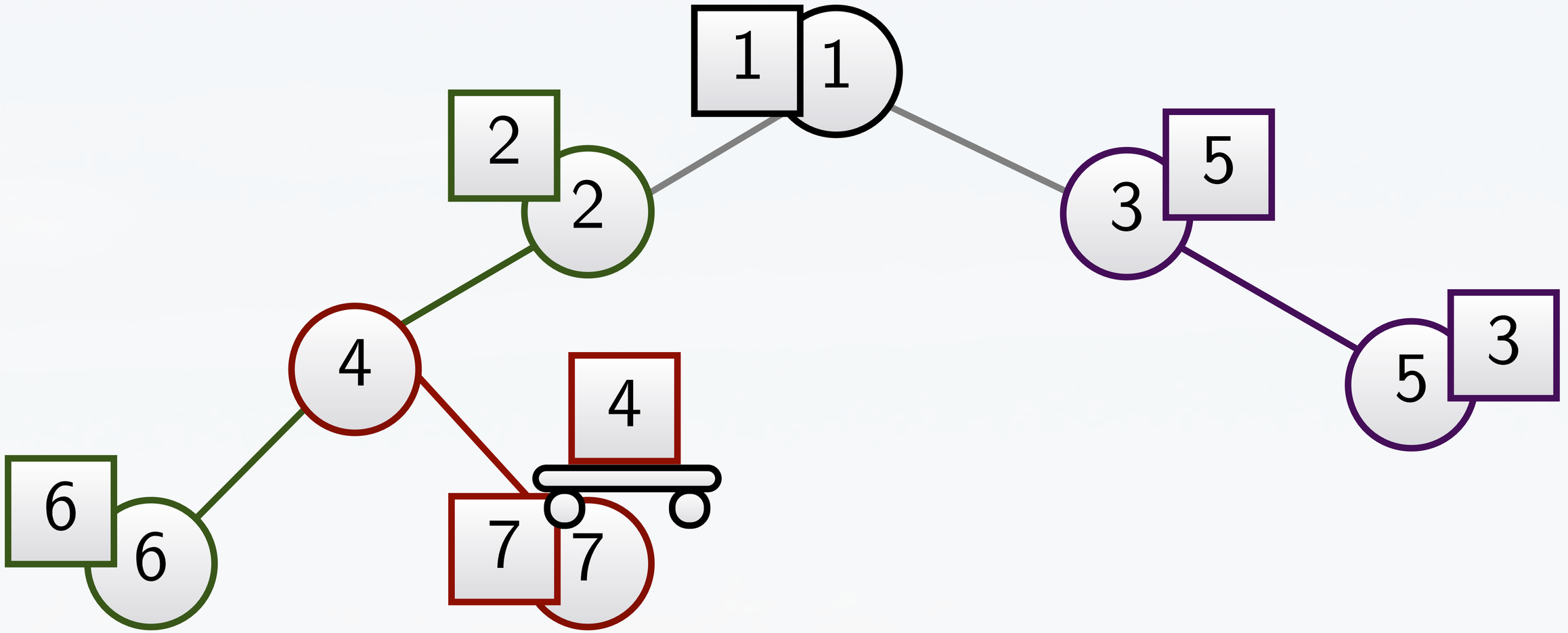

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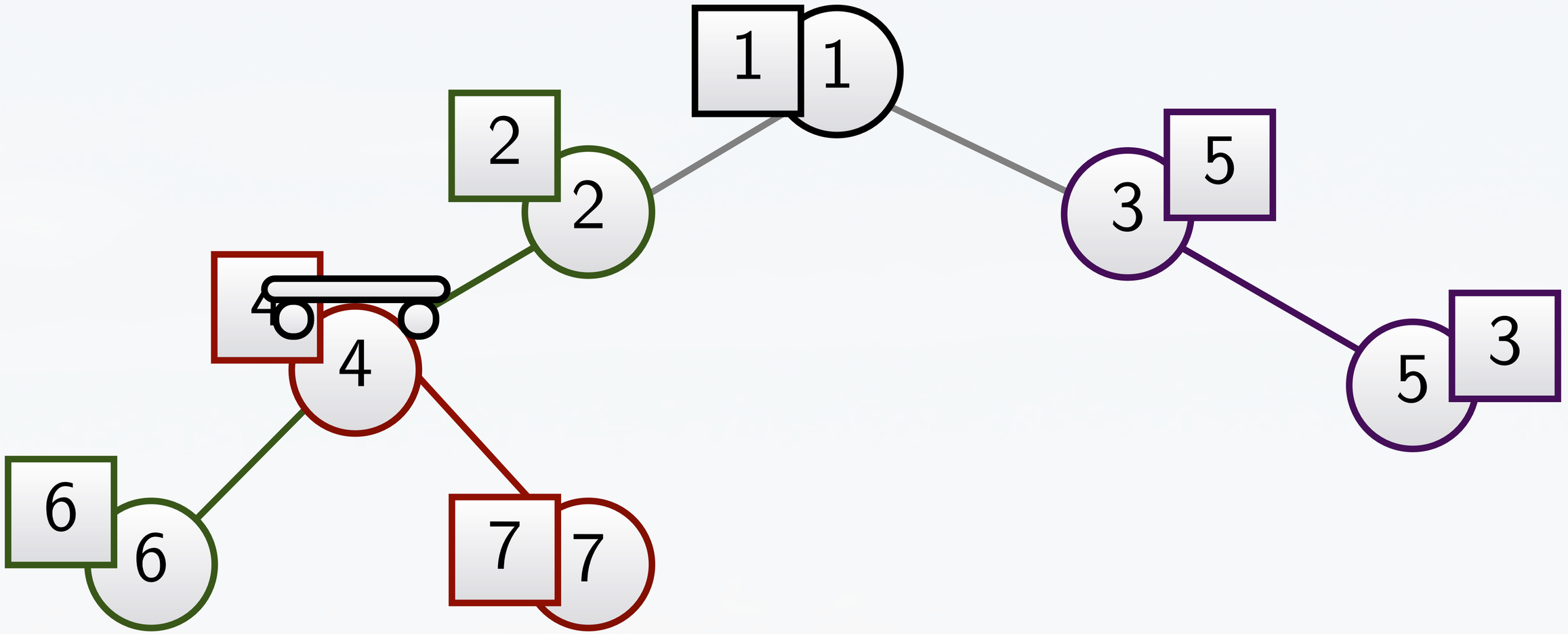

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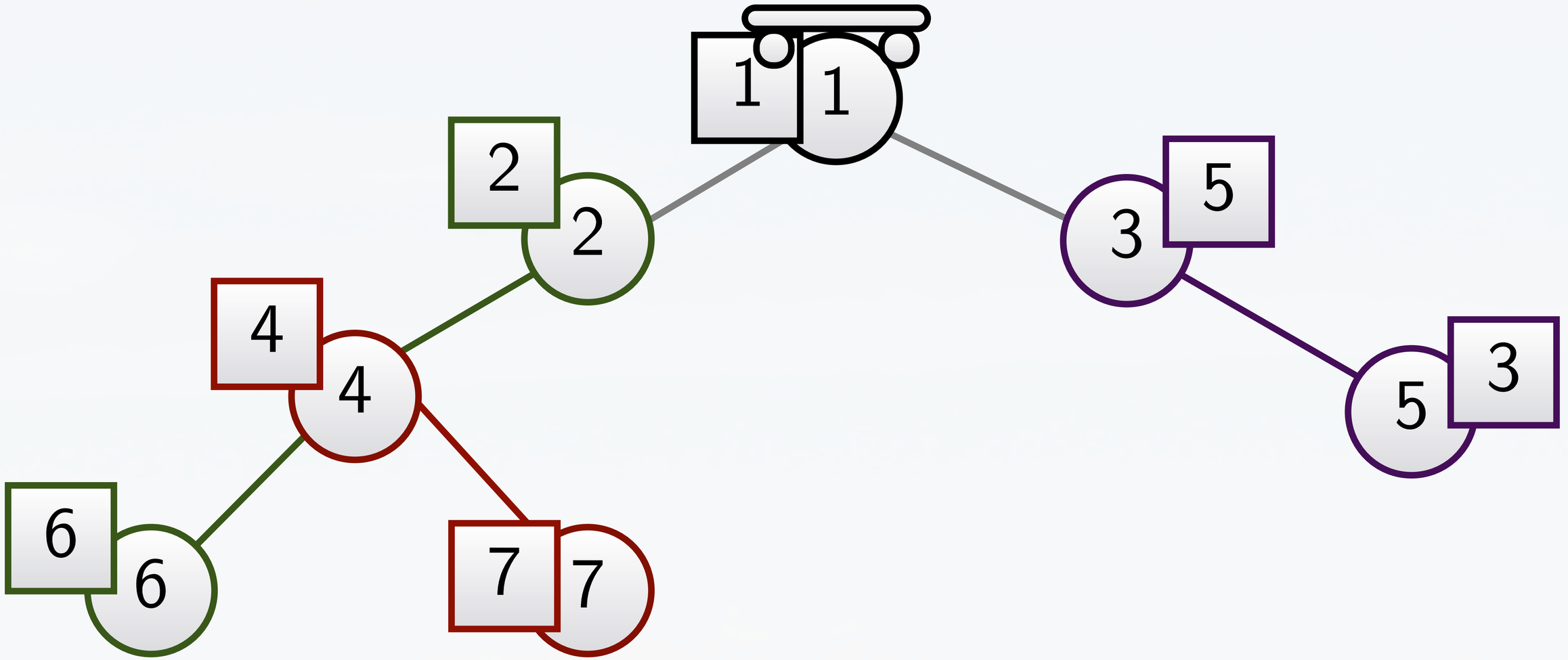

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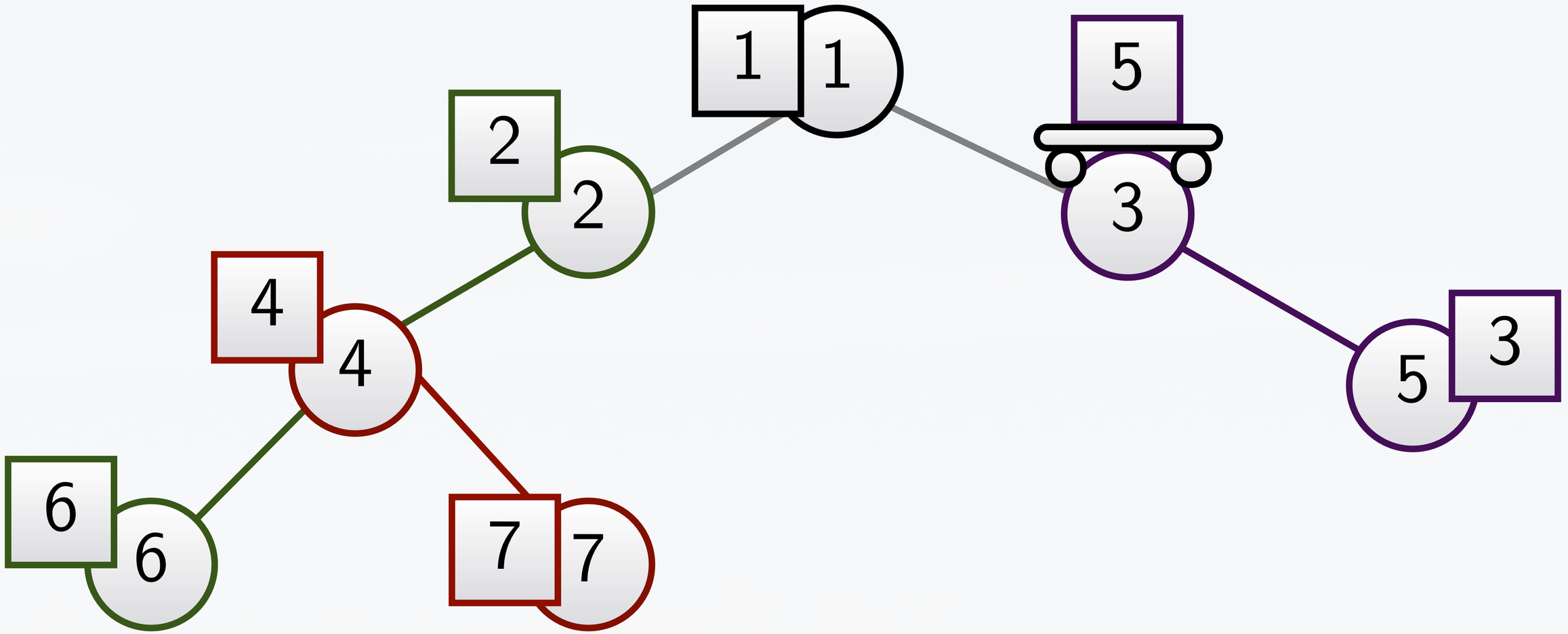

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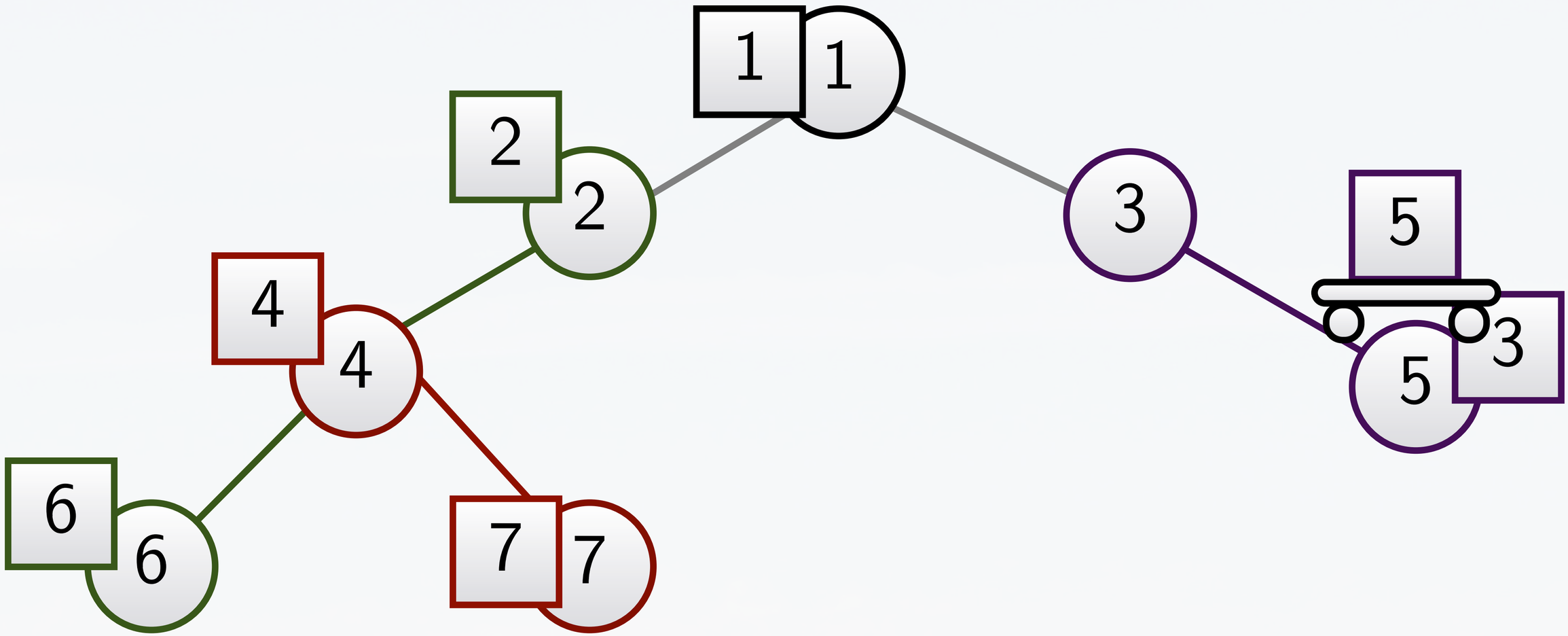
# Fewest Swaps

3. Sorting
3.3 Swap-Count



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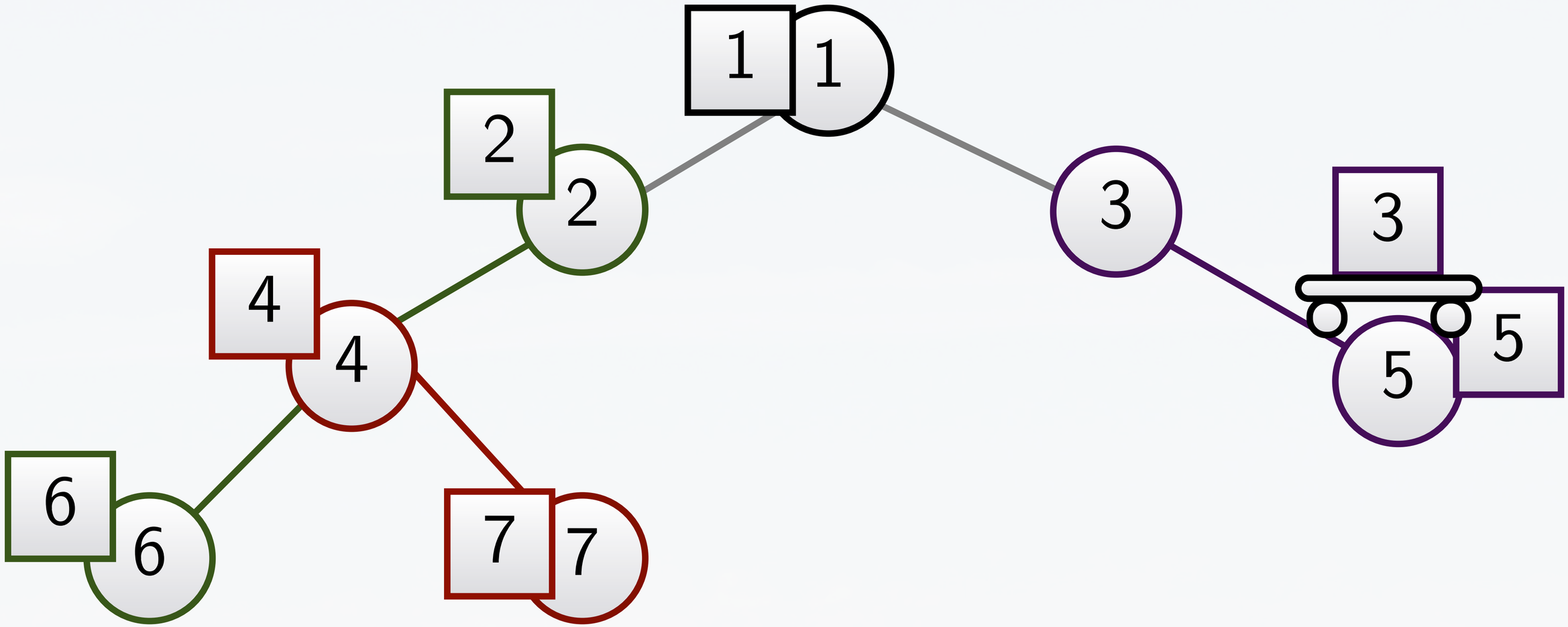

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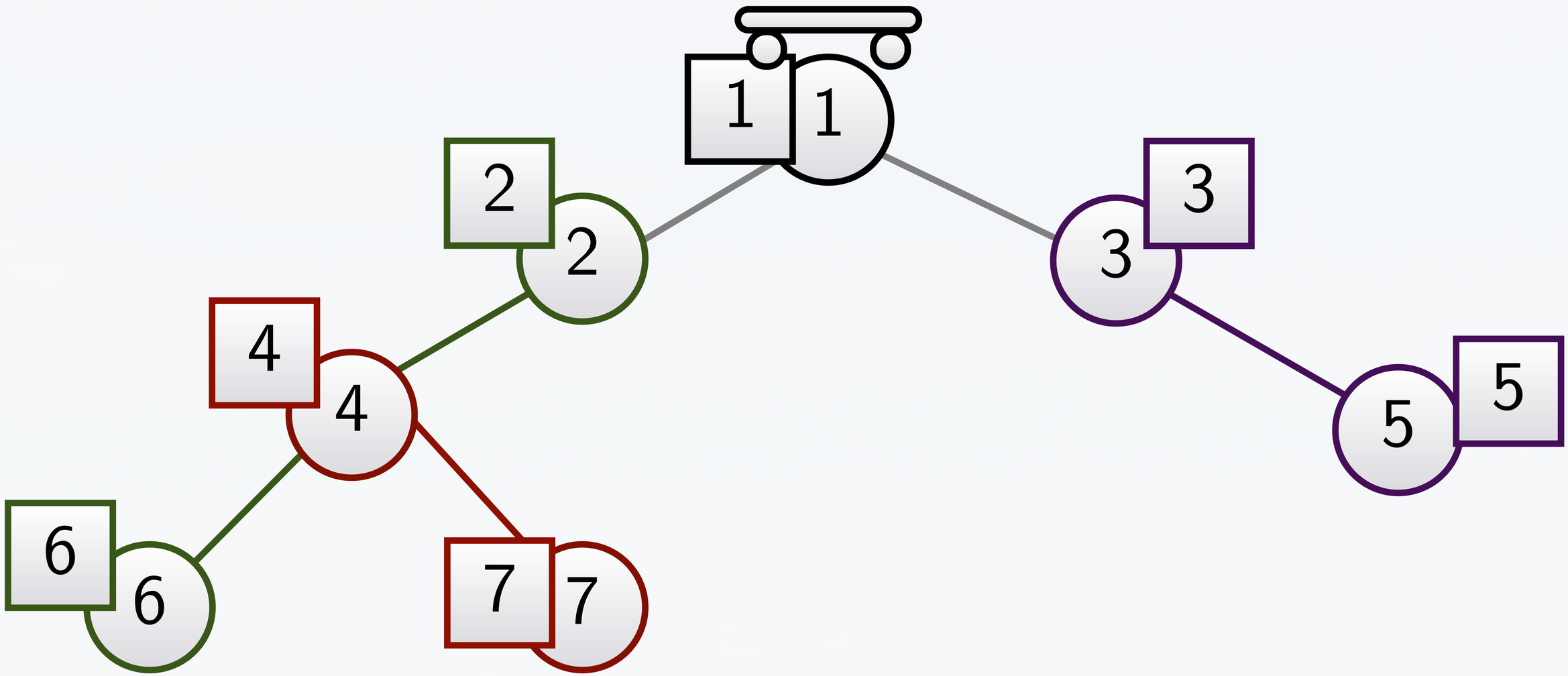

# Fewest Swaps



*Theorem:* Finding a swap-optimal sorting walk on a tree is NP-complete.

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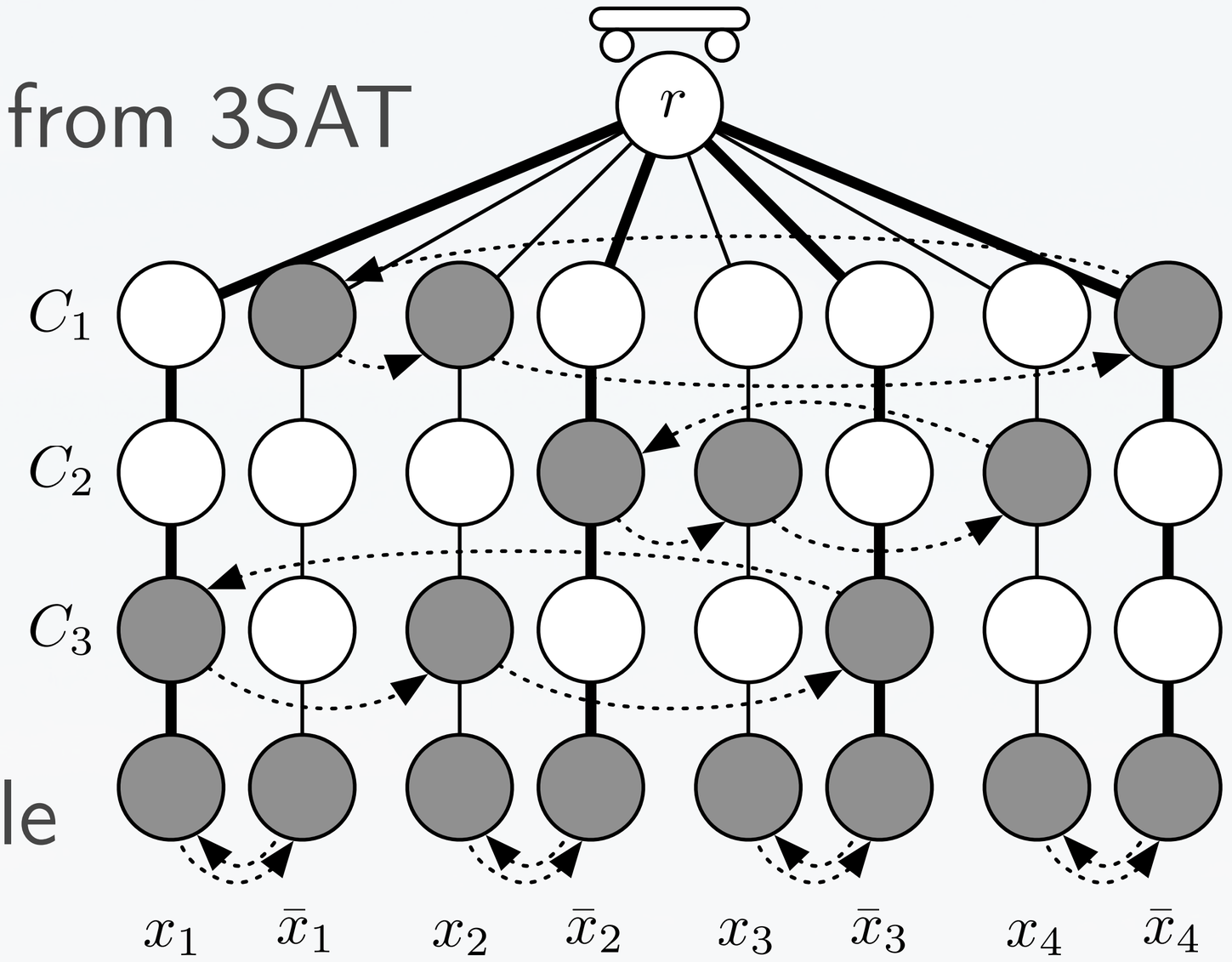
# Fewest Swaps



*Theorem:* Finding a swap-optimal sorting walk on a tree is NP-complete.

*Proof:* Reduction from 3SAT

- Spider graph
- One leg per literal
- One cycle per clause & variable



# Related Work



Sorting of physical objects

- using prefix reversals, stacks and queues, etc.

Sorting using a given set of possible swaps

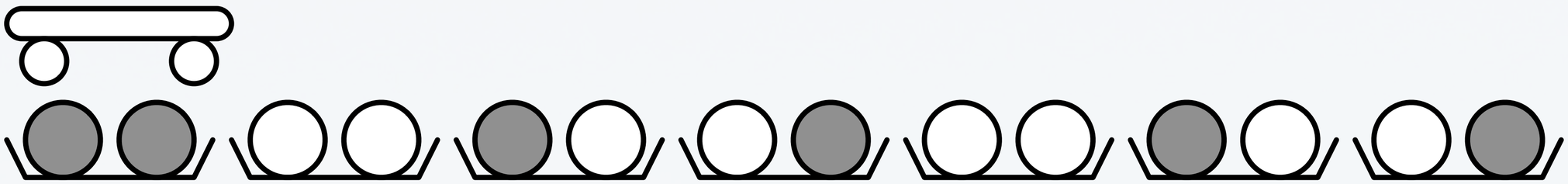
- swapping any pair [1849, Cayley]
- swapping adjacent pairs [1973, Knuth]
  - bubble sort is optimal (inversion number)
  - 2-approximation on trees [2014, Yamanaka et al.]

Group Steiner Tree Problem [1999, Ihler et al.]

# Task Kutije



Coding competition task by Luka Kalinovcic, 2006



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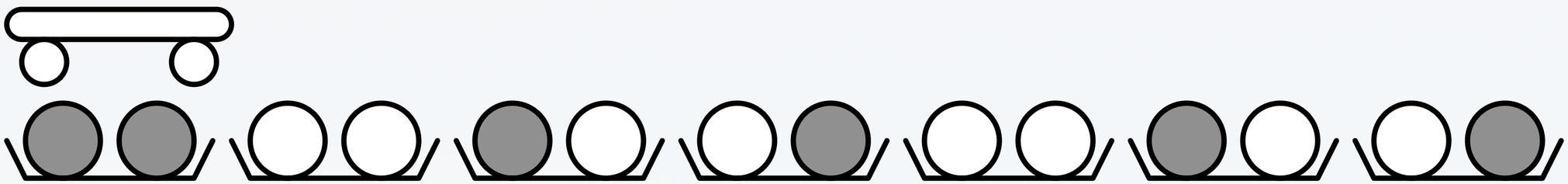
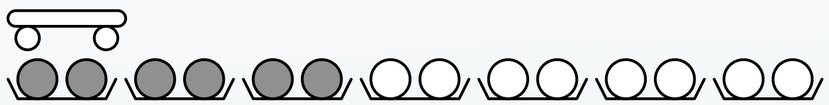
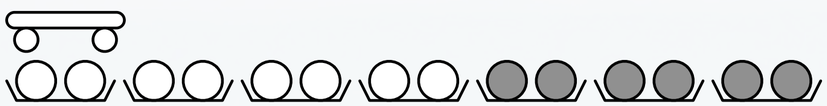


Robot that can carry up to two marbles at a time.

*Goal:*  or 

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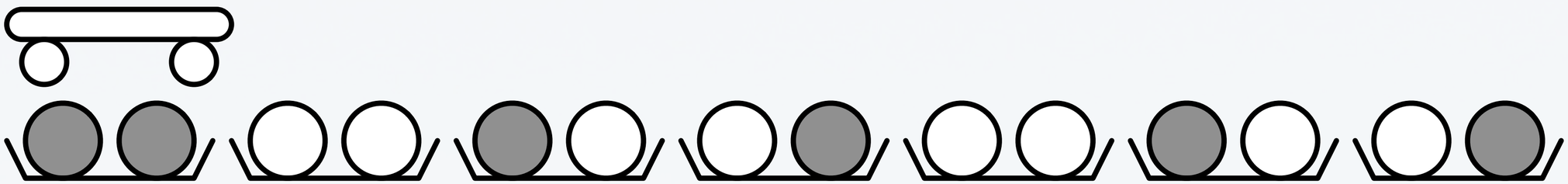

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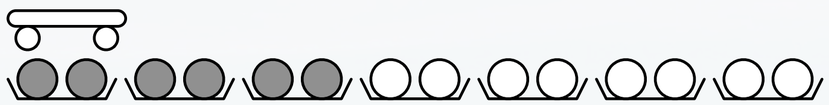
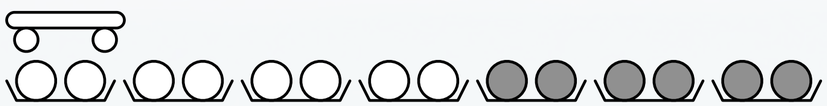
*Goal:*  or 

*Solution:* Build graph of lower-bound constraints. Find an Eulerian path through it. Use an essential-step-argument to show optimality.

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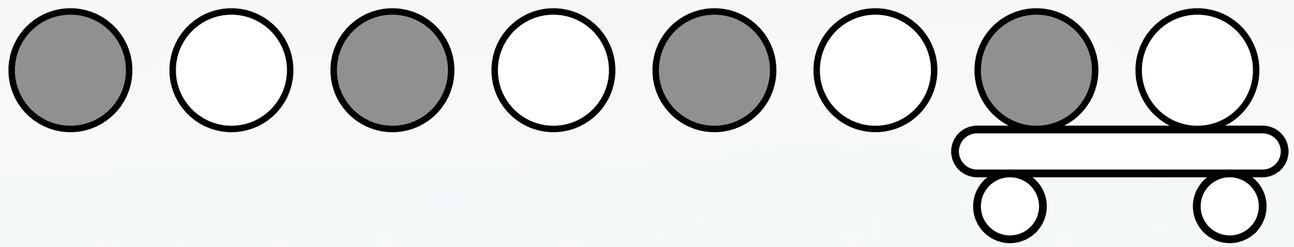
# Tait's Puzzle



Coin Puzzle by Peter Tait (1890), ICPC WF 2014

In every move this robot can carry exactly two neighboring marbles for an arbitrary distance.



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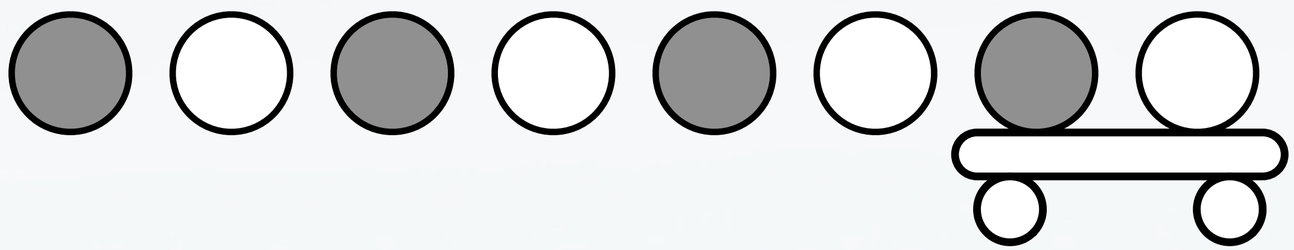

*Solution:* Show inductively that it is always possible to do in $n$ steps for $n$ pairs of marbles.

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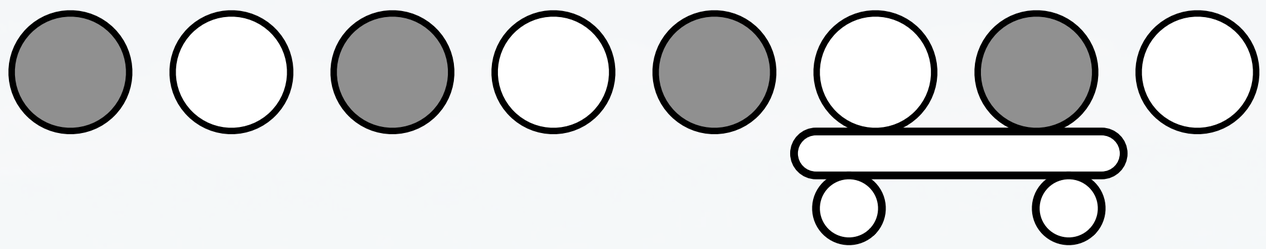

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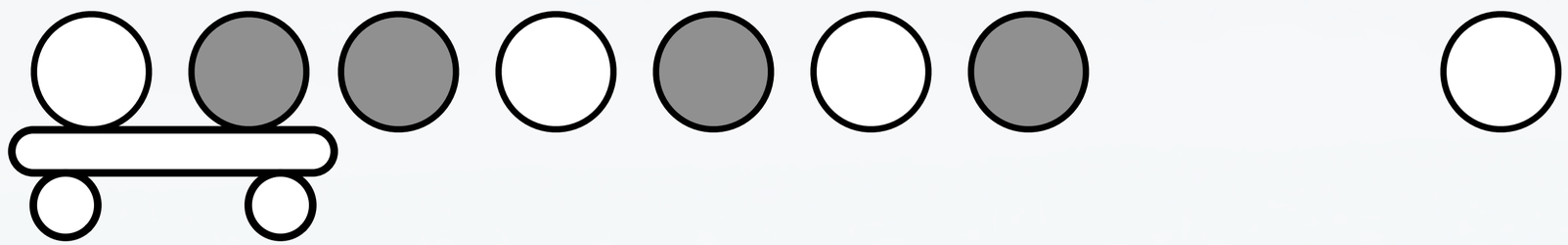

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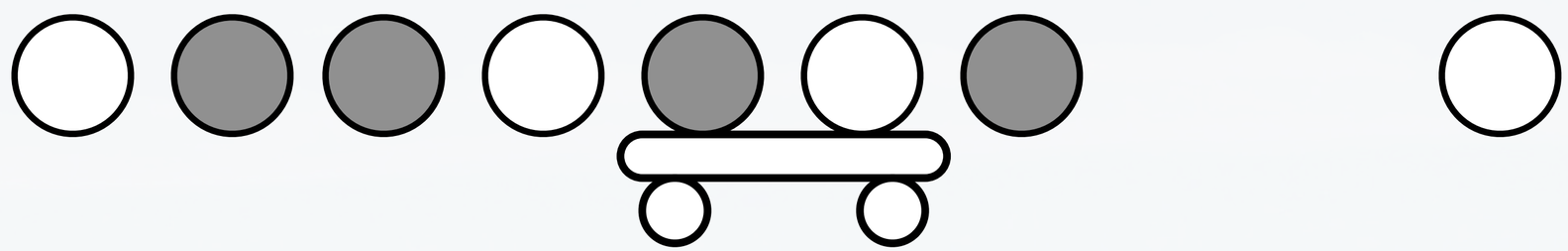

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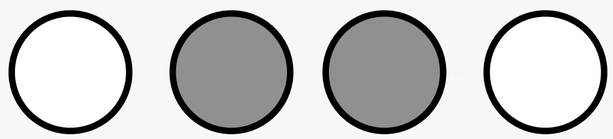

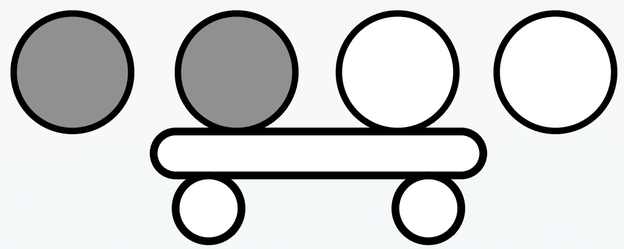

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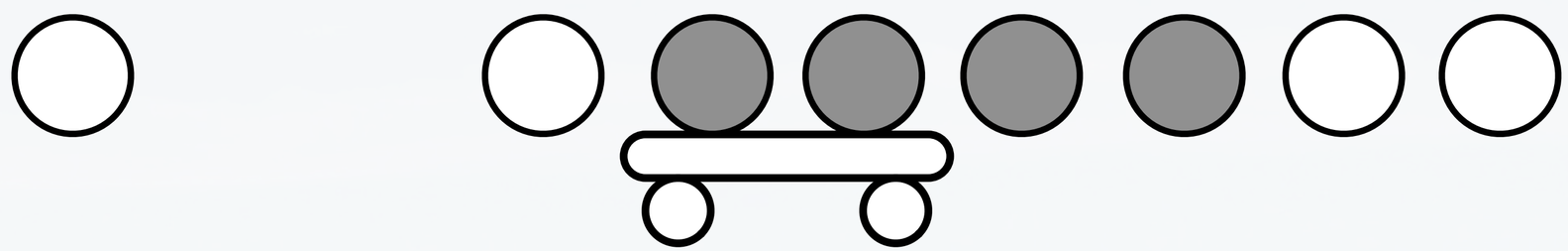

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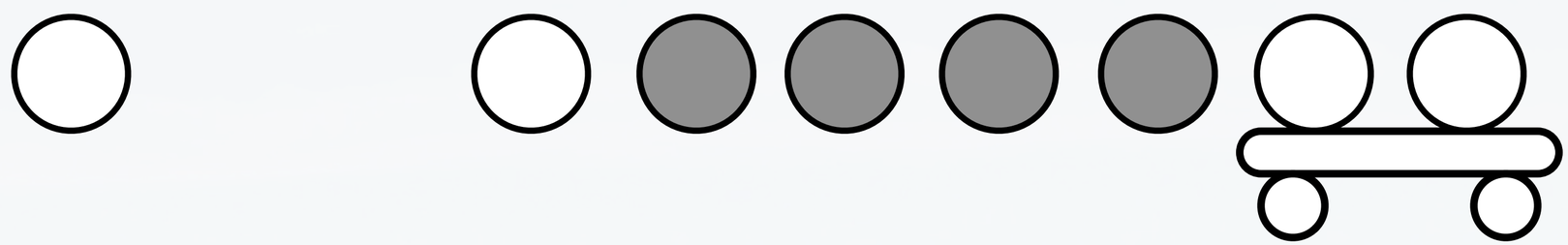

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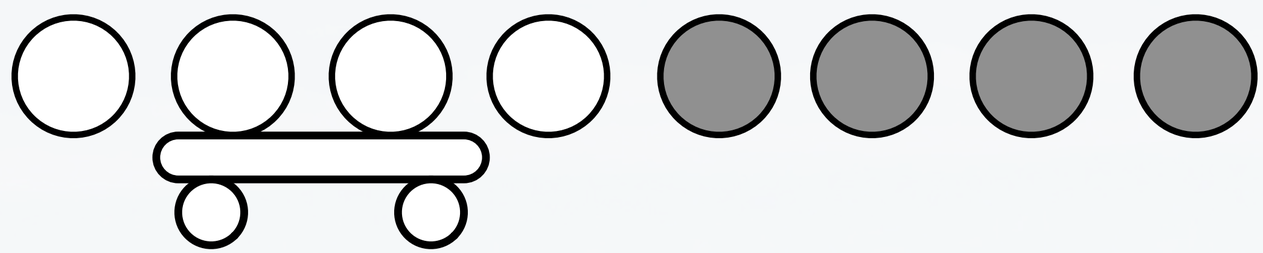

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# Other Systems

## What else did people try for automated parking?

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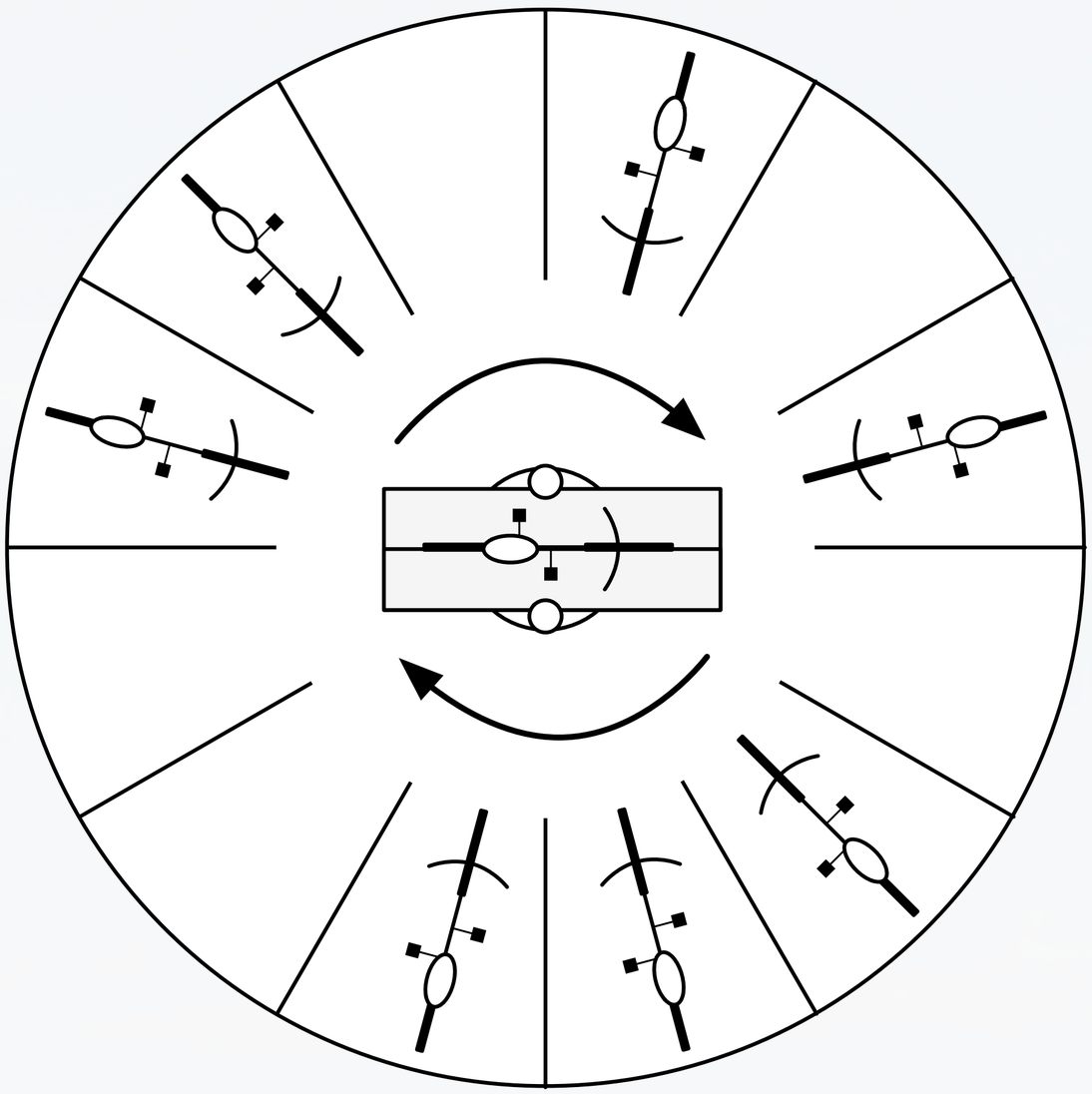
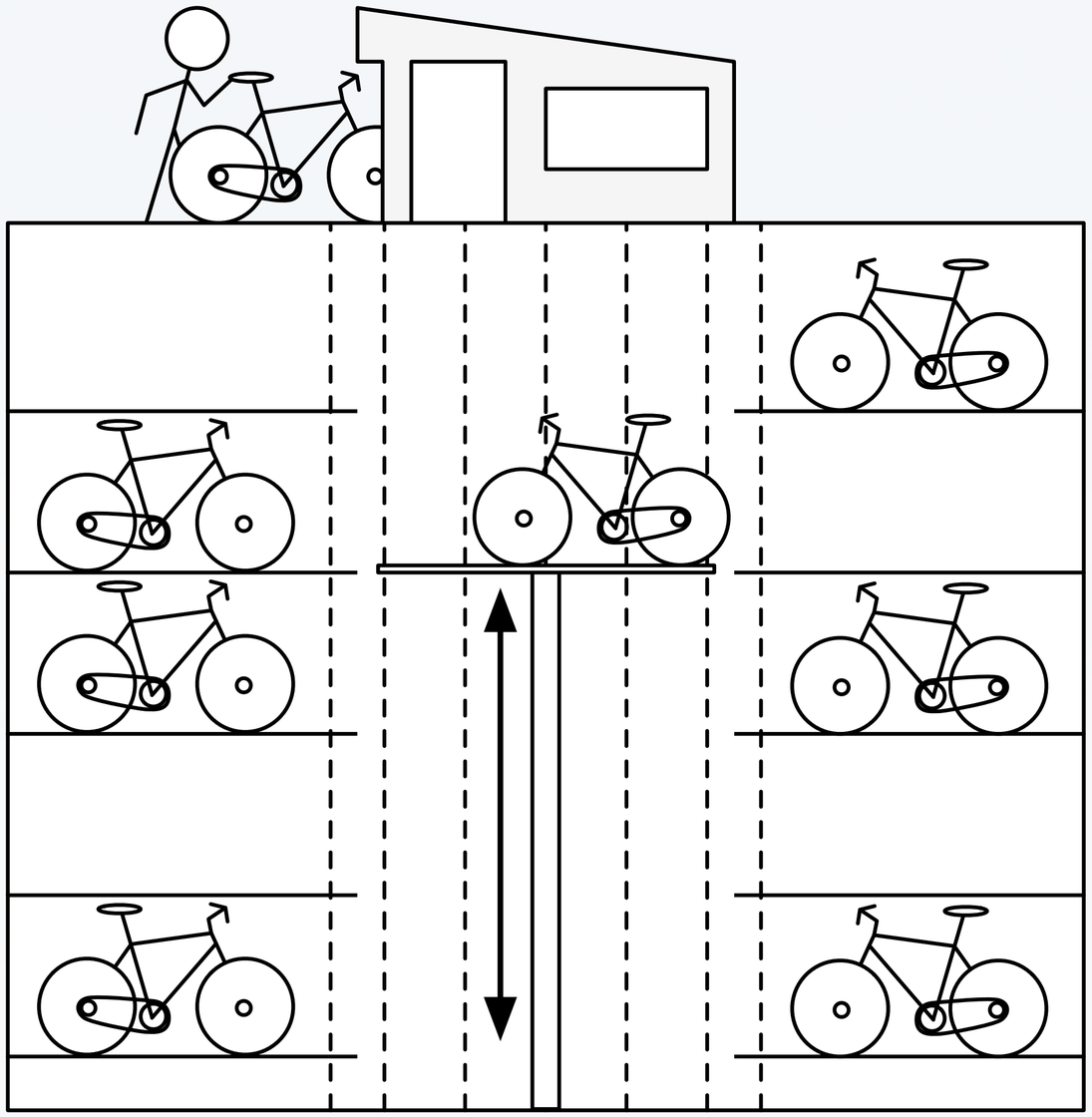
# Cylindrical



Vertical lift with circular storage slots around it





# ECO Cycle



## Example: ECO Cycle in Tokyo by Giken Seisakusho


Video by Giken Seisakusho Co. LTD.

# Züri West



## Example: Auto Züri West by Skyline Parking


Video by Skyline Parking AG

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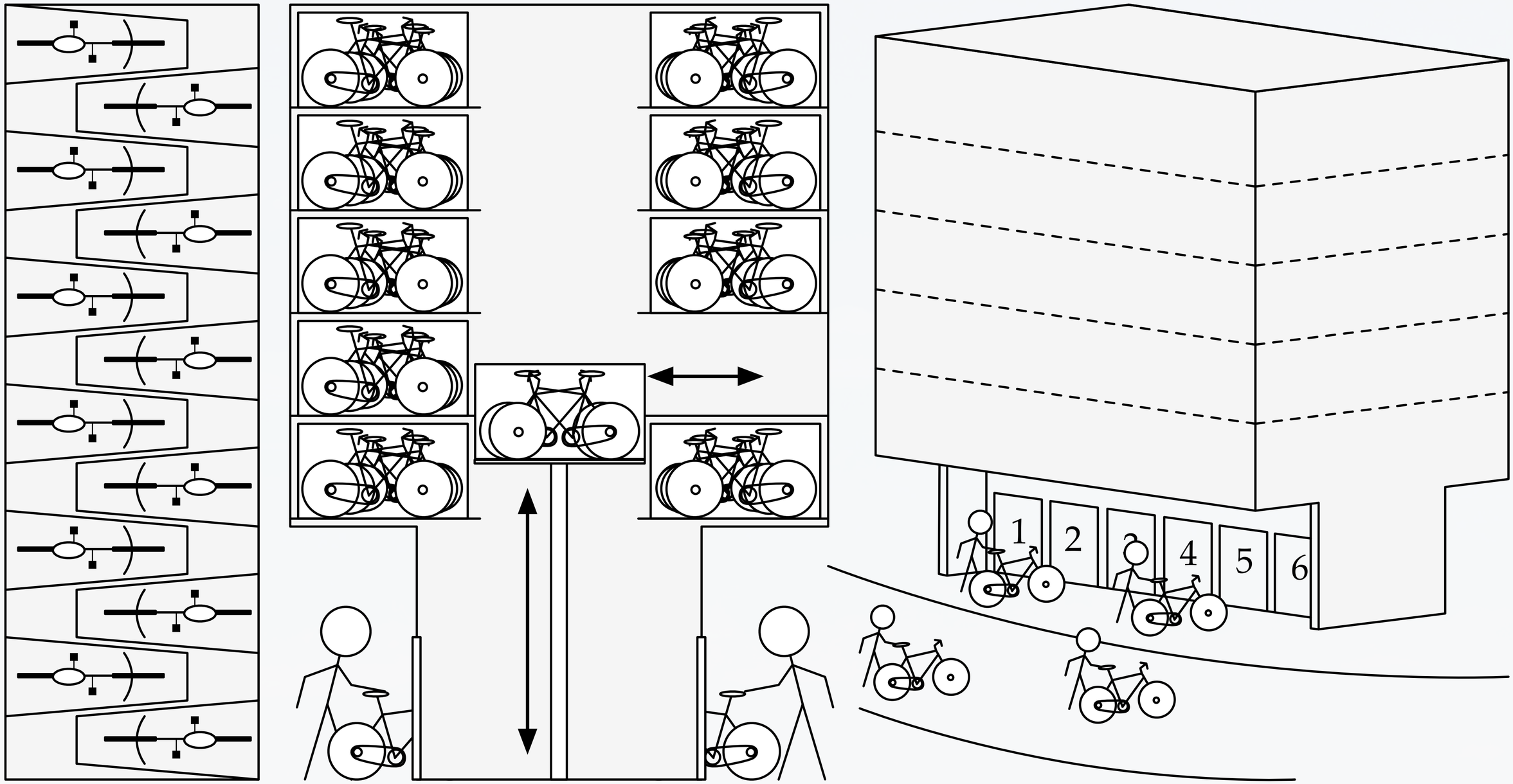
# Pallet Based



## Vertical lift for pallets of 12 bicycles



# Radhaus



## Example: Radhaus in Offenburg (D)

# Radhaus



## Example: Radhaus in Offenburg (D)

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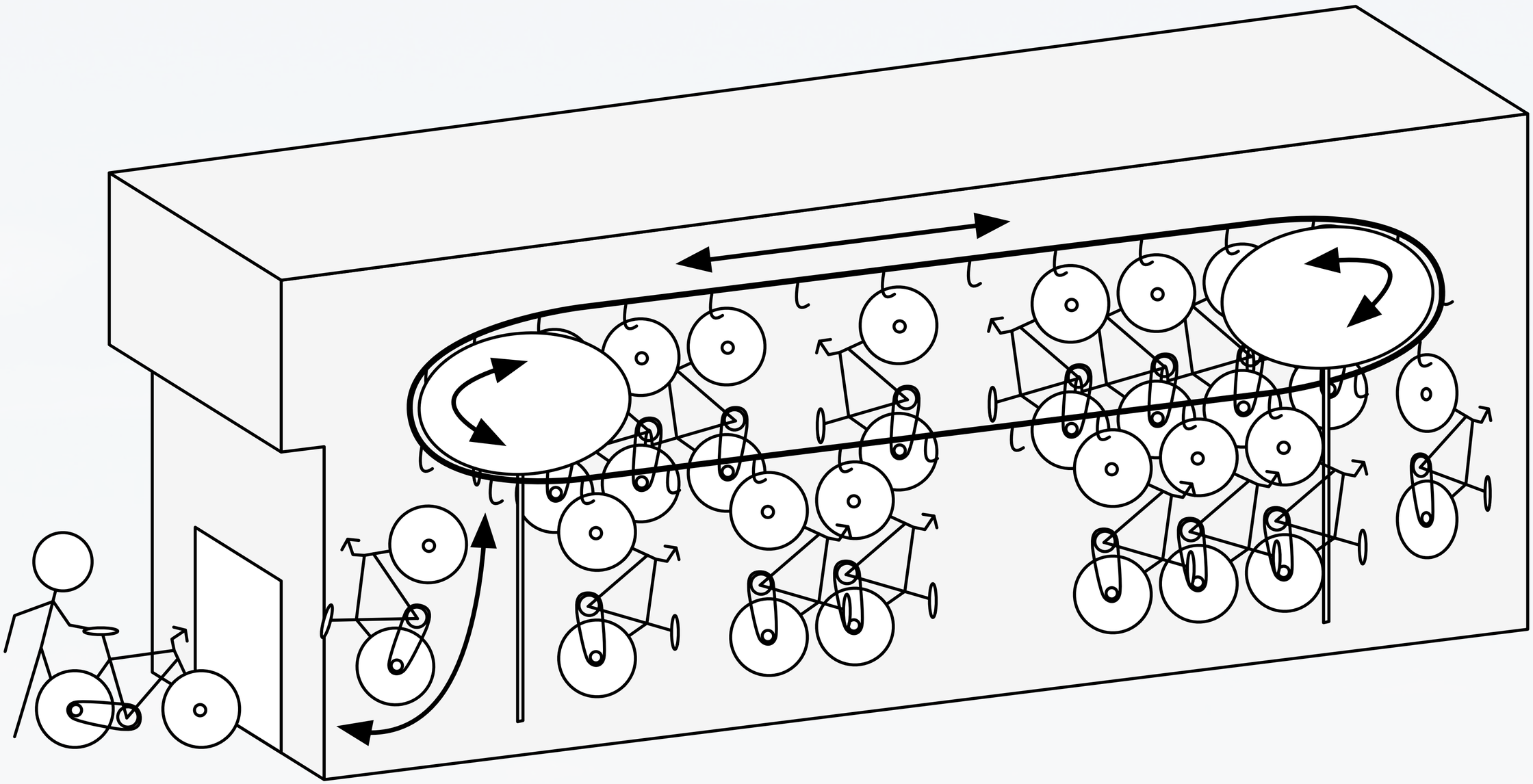
# Conveyor Belt



Cyclic chain of hooks, one central access point



# VeloMinck



## Example: VeloMinck in the Netherlands

Video by Lo Minck Systemen BV

# VeloMinck



## Example: VeloMinck in the Netherlands

Video by Lo Minck Systemen BV

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# Car Paternoster



## Example: Paternoster Car Parking in Chicago 1932

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# Conclusion



# Contributions



Scheduling Problem

- algorithms for arrival-only with a single door
- *NP*-hardness of the two-door problem
- *NP*-hardness of the all-day problem

Sorting Problem

- algorithms for sorting on paths, trees and cycles
- *NP*-hardness for planar graphs
- *NP*-hardness for trees when minimizing swaps

# Future Work



Open research questions

- approximation algorithms for two-door setting
- sorting problem on other graph classes
- further robot variations, multiple robots
- testing these models with real usage data

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# Questions



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# Thank you for your attention!

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